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Chapter 1

Introduction

1.1 From the Authors

This book is a work in progress. Please forgive (or point out) any mistakes you see. Feel free to contact the authors of any of these notes. Feel free to use these notes in your classes.

Texas-style teaching of mathematics courses was developed and practiced by Robert Lee Moore throughout a mathematical career spanning from 1915 until 1969. The method has been advocated by such veteran mathematicians as Halmos [13], Jones [16], and Yorke [35]. Dr. Hubert Stanley Wall worked in conjunction with Moore from the mid-1940's until the late 1960's and Hyman Joseph Ettlinger associated with Moore from 1920 to 1970. Throughout the years mathematicians have adopted and modified Moore's original method (and notes) for use in a variety of mathematics courses. Perhaps the best references to both the man and the method are the Mathematical Association of America's documentary on Moore [23] the texts that Wall developed while at the University of Texas, [32-33], and the biography of Moore developed by D. Redginald Traylor, [29].

This is a book of materials on courses modeled after the Moore Method. The conclusion of the book offers further discusson of the method along with references which are described in some detail for the reader wishing to familiarize himself with other aspects of the method. Suffice it to say that the method is "discovery" driven and, in the words of Redge Traylor [29] "encourages the students to do research at his own level."

This book provides a starting point for instructors interested in using such methods in the classroom and serves as a reference for those who are already familiar with these techniques. Texas-style methods are already widely implemented in mathematical curricula across the nation, however, there are few resourses and support materials available to those who wish to use this method of teaching. For this reason, instructors desiring to implement a Texas-style course often adapt notes from courses they took as students or develop their own notes. Both of these activities are time consuming and difficult to implement smoothly on the first attempt. This book provides theorem sequences for a variety of undergraduate and graduate courses which might be used as a starting point. Each theorem sequence constitutes a self-contained course that has been successfully implemented, often for years, by the contributor of that sequence. In addition to the theorem sequences, each contributor provides some helpful comments about his experiences using the sequence.

We are deeply indebted to all of those who have supported our efforts by contributing their own materials, proof reading, and offering support in other ways. A brief list of those individuals include: TBA.

1.2 From Advocates of the Method

> What then is the secret–what is the best way to learn to solve problems? The answer is implied by the sentence I started with: solve problems. The method I advocate is sometimes known as the 'Moore method', because R.L. Moore developed and used it at the University of Texas. It is a method of teaching, a method of creating the problem-solving attitude in a student, that is a mixture of what Socrates taught us and the fiercely competitive spirit of the Olympic games. *P. R. Halmos*

> What Moore did: ... After stating the axioms and giving motivating examples to illustrate their meaning he would then state definitions and theorems. He simply read them from his book as the students copied them down. He would then instruct the class to find proofs of their own and also to construct examples to show that the hypotheses of the theorems could not be weakened, omitted, or partially omitted. ... "When a student stated that he could prove Theorem x, he was asked to go to the blackboard and present the proof. Then the other students, especially those who hadn't been able to discover a proof, would make sure that the proof presented was correct and convincing. Moore sternly prevented heckling. This was seldom necessary because the whole atmosphere was one of a serious community effort to understand the argument. *F.Burton Jones*

> Since the roots of the problems described above run so deep it is imperative that potential solutions (such as the Moore method) be implemented early in students' careers – and not just for stduents planning to become mathematicians. *J. A. Yorke and M. D. Hartl*

> *D. Taylor* gives the following criteria which characterize the Moore method of teaching include:
>
> - The fundamental purpose: that of causing a student to develop his power at rational thought.

- Collecting the students in classes with common mathematical knowledge, striking from membership of a class any student whose knowledge is too advanced over others in the class.
- Causing students to perform research at their level by confronting the class with impartially posed questions and conjectures which are at the limits of their capability.
- Allowing no collective effort on the part of the students inside or outside of class, and allowing the use of no source material.
- Calling on students for presentation of their efforts at settling questions raised, allowing a feeling of "ownership" of a theorem to develop.
- Fostering competition between students over the settling of questions raised.
- Developing skills of critical analysis among the class by burdening students therein with the assignment of "refereeing" an argument presented.
- Pacing the class to best develop the talent among its membership.
- Burdening the instructor with the obligation to not assist, yet respond to incorrect statements, or discussions arising from incorrect statements, with immediate examples or logically sound propositions to make clear the objection or understanding.

Chapter 2

Analysis, Mahavier and Mahavier

2.1 To the Instructor

This theorem sequence constitutes a self-contained, one-semester course in real analysis. The goal of the course is two fold. First we desire to mature the student mathematically and secondly we prove rigorously the theorems that are typically touched upon in an introductory calculus course, but not rigorously proved, including the extreme value theorem, mean value theorem, Rolles theorem, the fundamental theorem of calculus, etc. Because the notes require no prerequisites they can serve as a replacement or a sequel to courses commonly referred to as "Foundations of Mathematics" or "A First Course in Logic."

This course is structured similarly to the course entitled, *A first course in topology*, and the reader should read the introduction to that sequence for a discussion of the format, sylabus, and grading for the course.

A problem or theorem marked (T) is topological in nature. A problem or theorem marked (CA) requires the completeness axiom. A problem or theorem marked (C) is related to Cauchy sequences and the material following does not require the use of this problem or theorem.

2.2 To the Student

Analysis is an area of mathematics, just as Algebra, Geometry, and Topology are areas of mathematics and is usually defined heursitically or not at all. In your calculus sequence you learned about the topics of limits, continuity, differentiability, and integration at an introductory level. In this course, we will follow the same order that you followed in calculus, but we will spend more time on the mathematical structures than on the application of the concepts. We will define each concept rigorously and present the material that you will recognize from calculus such as the extreme value theorem, mean value theorem, Rolles theorem, and the fundamental theorem of calculus.

2.3 Theorem Sequence

Definition 1 *By a* **point** *is meant an element of the real numbers, $\Re$.*

Definition 2 *By a* **point set** *is meant a collection of one or more points.*

Definition 3 *The statement that the point set M is* **linearly ordered** *means that there is a meaning for the words "less than ($<$)," "less than or equal to ($\leq$)," "greater than ($>$)," and "greater than or equal to ($\geq$)." If each of a, b and c is in M, then*

- *if $a \leq b$ and $b \leq c$ then $a \leq c$*
- *one and only one of the following is true:*
  - $a \leq b$,

- $b \leq a$, *or*
- $a = b$.

Axiom 4 $\Re$ *is linearly ordered.*

Axiom 5 *If p is a point there is a point less than p and a point greater than p.*

Axiom 6 *If p and q are two points then there is a point between them, for example, (p+q)/2.*

Axiom 7 *If* $a < b$ *and c is any point, then* $a + c < b + c$,

Axiom 8 *If* $a < b$ *and* $c > 0$, *then* $a \cdot c < b \cdot c$. *If* $c < 0$, *then* $a \cdot c > b \cdot c$.

Axiom 9 *If x is a point, then x is an integer or there is an integer n such that* $n < x < n + 1$.

Definition 10 *The statement that the point set O is an* **open interval** *means that there are two points a and b such that O is the set of all points between a and b.*

Definition 11 *The statement that I is a* **closed interval** *means that there are two points a and b such that* $p \in I$ *if and only if p=a, p=b, or p is between a and b.*

Notation: We use the notation (a,b) to denote the open interval consisting of all points p such that $a < p < b$. Similarly we use the notation [a,b] to denote the closed interval determined by the two points a and b where $a < b$. We do not use (a,b) or [a,b] in case $a = b$, although many mathematicians and texts do.

Definition 12 *If M is a point set and p is a point, the statement that p is a* **limit point** *of the point set M means that* every *open interval containing p contains a point of M different from p.*

Problem 13 *Show that if M is the open interval* (a, b), *and p is in M, then p is a limit point of M.*

Problem 14 *Show that if M is the closed interval* $[a, b]$, *and p is not in M, then p is not a limit point of M.*

Problem 15 *Show that if M is a point set having a limit point, then M contains (at least) 2 points.*

Problem 16 *Show that if M is the set of all positive integers, then no point is a limit point of M.*

Question 17 *Assume M is a point set such that if p is a point of M, there is a first point of M to the left of p and a first point of M to the right of p. Is it true that M cannot have a limit point?*

Problem 18 *Show that if H is a point set and K is a point set and p is a limit point of $H \cap K$, then p is a limit point of H and p is a limit point of K.*

Problem 19 *Show that if H is a point set and K is a point set and every point of K is a limit point of H and p is a limit point of K, then p is a limit point of H.*

Problem 20 *If H is a point set and K is a point set and p is a limit point of $H \cup K$, then p is a limit point of H or p is a limit point of K.*

Problem 21 *Show that if M is the set of all reciprocals of positive integers, then 0 is a limit point of M.*

Definition 22 *The statement that the point sequence $p_1, p_2, \ldots$* **converges** *to the point p means that if S is an open interval containing p then there is a positive integer n such that if m is a positive integer and $m > n$ then $p_m \in S$.*

Definition 23 *The statement that the sequence $p_1, p_2, p_3, \ldots$* **converges**, *means that there is a point p such that $p_1, p_2, p_3, \ldots$ converges to p.*

Problem 24 *For each positive integer n, let $p_n = 1 - 1/n$. Show that the sequence $p_1, p_2, p_3, \ldots$ converges to 1.*

Problem 25 *For each positive integer n, Let $p_{2n-1} = 1/(2n-1)$ and let $p_{2n} = 1 + 1/2n$. Show that the sequence $p_1, p_2, p_3, \ldots$ does not converge to 0. Hint: If m is a positive, odd integer then $p_m = 1/m$ while if m is a positive, even integer then $p_m = (m+1)/m$.*

Problem 26 *For each positive integer n, let $p_{2n} = 1/(2n-1)$, and let $p_{2n-1} = 1/2n$. Show that the sequence $p_1, p_2, p_3, \ldots$ converges to 0.*

Notation: If $p_1, p_2, p_3, \ldots$ is a sequence, then $\{p_i\}$ denotes the **range** of the sequence. That is, $\{p_i\}$ denotes the point set to which the point x belongs if and only if there is a positive integer n such that $x = p_n$.

Problem 27 *Show that if the sequence $p_1, p_2, p_3, \ldots$ converges to the point p, and, for each positive integer n, $p_n \neq p_{n+1}$, then p is a limit point of the set which is the range of the sequence.*

Problem 28 *Show that if $p \neq 0$, then p is not a limit point of the set $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$.*

Problem 29 *Show that if c is a number and $p_1, p_2, p_3, \ldots$ is a sequence which converges to the point p, then the sequence $c \cdot p_1, c \cdot p_2, c \cdot p_3, \ldots$ converges to $c \cdot p$.*

Problem 30 *Show that if the sequence $p_1, p_2, p_3, \ldots$ converges to p and the sequence $q_1, q_2, q_3, \ldots$ converges to q, then the sequence $p_1 + q_1, p_2 + q_2, p_3 + q_3, \ldots$ converges to $p + q$.*

Definition 31 *The statement that p is the* **first point to the right of the point set M** *means that p is to the right of every point of M and if q is a point to the left of p, then q is not to the right of every point of M.*

Definition 32 *The statement that p is the* **rightmost point of M** *means that p is in M and no point of M is to the right of p.*

Definition 33 **Leftmost point of** *M and* **first point to the left of** *M are defined similarly.*

Problem 34 *Show that if M is a point set, there cannot be a both a rightmost point of M and a first point to the right the point set M.*

Problem 35 *Show that if M is a point set and there is a point p which is the first point to the right of M, then p is a limit point of M.*

Theorem 36 *If the sequence $p_1, p_2, p_3 \ldots$ converges to the point p and q is a point different from p, then $p_1, p_2, p_3, \ldots$ does not converge to q.*

Definition 37 *The statement that the point set M is* **finite** *means that there is a positive integer n such that M contains n points but M does not contain $n + 1$ points.*

Definition 38 *The statement that the point set M is* **infinite** *means that M is not finite.*

Theorem 39 *If M is a finite point set then M has a rightmost point and a leftmost point.*

Theorem 40 *If the point p is a limit point of the point set M and S is an open interval containing p, then $S \cap M$ is infinite.*

Theorem 41 *If the sequence $p_1, p_2, p_3, \ldots$ converges to the point p and q is a point different from p, then q is not a limit point of the range $\{p_i\}$ of the sequence $p_1, p_2, p_3, \ldots$.*

Definition 42 *(T) The statement that the point set M is an* **open** *point set means that for every point p of M there is an open interval which contains p and is a subset of M.*

Definition 43 *(T) The statement that the point set M is a* **closed** *point set means that if p is a limit point of M, then p is in M.*

Note: This does not mean that M has a limit point. If a set M has no limit point, then it is a closed point set. We could equivalently define closed by saying that M is closed if and only if there is no limit point of M that is not in M.

Theorem 44 *(T) If M is a closed point set, then the set of all points not in M is an open point set.*

Theorem 45 *(T) If M is an open point set, then the set of all points not in M is a closed point set.*

Theorem 46 *(T) If p is a point, there is a sequence of open intervals $S_1, S_2, S_3, \ldots$ each containing p such that for each positive integer n, $S_{n+1} \subseteq S_n$ and p is the only point that is in every open interval in the sequence.*

Definition 47 *The statement that the point set M is* **bounded** *means that M is a subset of a closed interval.*

Definition 48 *Let M be a point set. The statement that M is* **bounded below** *means that there is a point z such that z is less than or equal to m for every m in M.* **Bounded above** *is defined similarly.*

Theorem 49 *If the sequence $p_1, p_2, p_3, \ldots$ converges to the point p, then $M = \{p_1, p_2, p_3 \ldots\}$ is bounded.*

Axiom 50 The Completeness Axiom *If M is a point set and there is a point a to the right of every point of M, then there is either a rightmost point of M or a first point to the right of M.*

Notation: Similarly, if there is a point to the left of every point of M, then there is either left most point of M or a first point to the left of M.

Theorem 51 *(CA) If M is a closed and bounded point set, then there is a leftmost point of M and a rightmost point of M.*

Definition 52 *The statement that the sequence $p_1, p_2, p_3, \ldots$ is an* **increasing** *sequence means that for each positive integer n, $p_n < p_{n+1}$.*

Definition 53 *The statement that the sequence $p_1, p_2, p_3, \ldots$ is* **non-decreasing** *means that for each positive integer n, $p_n \leq p_{n+1}$.*

Definition 54 *We define a* **decreasing** *and* **non-increasing** *sequence similarly.*

Theorem 55 *(CA) If $p_1, p_2, p_3, \ldots$ is a non-decreasing sequence and there is a point, p, to the right of each point of the sequence, then the sequence converges to some point.*

Problem 56 *(T) Show that if M is a pointset and p is a point of M and every closed interval containing p contains a point of M different from p, then p is a limit point of M.*

Problem 57 *(T) Show that it is not true that if p is a limit point of a point set M, then every closed interval containing p must contain a point of M different from p.*

Definition 58 *(C) The statement that the sequence* $p_1, p_2, p_3, \ldots$ *is a* **Cauchy sequence** *means that if* ϵ *is a positive number, then there is a positive integer* n *such that if* m *is a positive integer and* k *is a positive integer,* $m \geq n$*, and* $k \geq n$*, then the distance from* p_m *to* p_k *is less than* ϵ*.*

Theorem 59 *(C)* $p_1, p_2, p_3, \ldots$ *is a Cauchy sequence if and only if it is true that for each positive number d, there is a positive integer n such that if m is a positive integer and* $m \geq n$*, then* $|p_m - p| < d$*.*

Theorem 60 *(C) If the sequence* $p_1, p_2, p_3, \ldots$ *converges to a point p, then* $p_1, p_2, p_3, \ldots$ *is a Cauchy sequence.*

Theorem 61 *(C) If* $p_1, p_2, p_3, \ldots$ *is a Cauchy sequence, then the set* $\{p_1, p_2, p_3, \ldots\}$ *is bounded.*

Theorem 62 *(C) If* $p_1, p_2, p_3, \ldots$ *is a Cauchy sequence, then the set* $\{p_1, p_2, p_3, \ldots\}$ *does not have two limit points.*

Theorem 63 *(C) If M is an infinite and bounded set of points then M has a limit point.*

Theorem 64 *(C) If* $p_1, p_2, p_3, \ldots$ *is a Cauchy sequence, then the sequence* $p_1, p_2, p_3, \ldots$ *converges.*

Notation: We now extend our definiton of the word, "point" to include points in the plane. We must determine from the context whether "point" means a point on the real line or a point in the plane. We will also have to determine from context whether the notation (x,y) denotes an open interval or the point with coordinates x and y.

Definition 65 *The statement that f is a* **function** *means that f is a set of points (x,y) in a plane such that no vertical line contains two of them.*

Definition 66 *If f is a function, then by the* **domain** *of f is meant the set of all first coordinates of points of f, and by the* **range** *of f is meant the set of all second coordinates of points of f.*

Notation: We use the usual notation that if x is a number in the domain of f, then f(x) is the number which is the 2nd coordinate of the point of f whose first coordinate is x. In other words, f(x) is the number such that (x,f(x)) is the point of f having x as its 1st coordinate.

Definition 67 *The statement the function f is* **continuous** *at the point* $p = (x, f(x))$ *means that*
i) p is a point on f, and
ii) if S is **any** *open interval containing the number f(x), then there is an open interval T containing the number x such that if* $t \in T$*, and t is in the domain of f, then* $f(t) \in S$*.*

Definition 68 *The statement that the function f is continuous at the number x means that x is in the domain of f and f is continuous at the point $(x, f(x))$.*

Definition 69 *The statement that* **f is a continuous function** *means that f is a function which is continuous at each of its points.*

Problem 70 *Let f be the function such that $f(x) = 2$ for all numbers $x > 5$, and $f(x) = 1$ for all numbers $x \leq 5$.*

- *Show that f is not continuous at the point (5,1).*
- *Show that f is continuous at the point (t,2) for each number $t > 5$.*

Problem 71 *Show that if f is a function and $(x, f(x))$ is a point on f, and x is not a limit point of the domain of f, then f is continuous at $(x, f(x))$.*

Problem 72 *Let f be the function such that $f(x) = x^2$ for all numbers x. Show that f is continuous at the point at the point (2,4).*

Problem 73 *If f is a function which is continuous on $[a, b]$ and $x \in [a, b]$ such that $f(x) > 0$ then there exists a open interval, T, containing x such that $f(t) > 0$ for all $t \in T$.*

Theorem 74 *If f is a function and $x_1, x_2, x_3, \ldots$ is a sequence of points in the domain of f converging to the number x in the domain of f, and f is continuous at $(x, f(x))$, then $f(x_1), f(x_2), \ldots$ converges to f(x).*

Definition 75 *If f and g are functions and there is a point common to the domain of f and the domain of g, then $f + g$ denotes the function h such that for each number x in the domain of both of f and g, $h(x) = f(x) + g(x)$.*

Theorem 76 *If f and g are functions and f is continuous at the point $(x, f(x))$ and g is continuous at the point $(x, g(x))$ and $h = f + g$, then h is continuous at the point $(x, h(x))$*

Theorem 77 *Suppose f and g are functions having domain M and each is continuous at the point p in M. Suppose that h is a function with domain M such that f(p)=h(p)=g(p) and for each number x in M, $f(x) \leq h(x) \leq g(x)$. Prove h is continuous at p.*

Theorem 78 *(CA,T) If $I_1, I_2, I_3, \ldots$ is a sequence of closed intervals such that for each positive integer n, $I_{n+1} \subseteq I_n$ then there is a point p such that if n is any positive integer, then p is in I_n. That is, there is a point p which is in all the closed intervals of the sequence $I_1, I_2, I_3, \ldots$.*

Theorem 79 *(CA,T) If $I_1, I_2, I_3, \ldots$ is a sequence of closed intervals and for each positive integer n, $I_{n+1} \subseteq I_n$, the length of I_n is less than $\frac{1}{n}$, then there is only one point p such that for each positive integer n, $p \in I_n$.*

Theorem 80 *If f is a continuous function whose domain includes the closed interval $[a,b]$, then the set of all numbers $x \in [a,b]$ such that $f(x) \geq 0$ is a closed point set.*

Theorem 81 *If f is a continuous function whose domain includes a closed interval $[a,b]$ and $p \in [a,b]$, then the set of all numbers $x \in [a,b]$ such that $f(x) = f(p)$ is a closed point set.*

Theorem 82 *(CA, T) No closed interval is the union of two mutually exclusive closed point sets.*

Problem 83 *(CA) If f is a function with domain the closed interval $[a,b]$ and the range of f is $\{-1,1\}$, then there is a number x in $[a,b]$ at which f is not continuous.*

Theorem 84 *(CA) If f is a continuous function whose domain includes a closed interval $[a,b]$ and $f(a) < 0$ and $f(b) > 0$, then there is a number x between a and b such that $f(x) = 0$.*

Theorem 85 *If f is a continuous function whose domain includes a closed interval [a,b], and L is a horizontal line, and (a,f(a)) is below L and (b,f(b)) is above L then there is a number x between a and b such that (x,f(x)) is on L.*

Definition 86 *The non-vertical line L is* **tangent to the function f at the point** *$P = (x,y)$ means that:*
i) x is a limit point of the domain of f,
ii) P is a point of L, and
iii) if A and B are non-vertical lines with the line L, except for P between them, then there are two vertical lines H and K such that if Q is a point of f between H and K which is not P, then Q is between A and B

Definition 87 *If f is a function the statement that f has a* **derivative** *at the number a in the domain of f means that f has a non-vertical tangent line at the point $(a, f(a))$. We use the notation $f'(a)$ to denote the slope of the line tangent to f at the point $(a, f(a))$ and $f'(a)$ is called the* **derivative of** *f* **at** *a.*

Definition 88 *If f is a function, the statement that f has* **derivative** *D at the number x in the domain of f means that*
i) x is a limit point of the domain of f,
ii) if (a,b) is an open interval containing D, then there is a open interval (h,k) containing x such that if t is a number in (h,k) and in the domain of f, and $t \neq x$, then

$$\frac{f(t) - f(x)}{t - x} \in (a,b)$$

.

As an alternative to this definition:

Definition 89 *If f is a function, the statement that f has* **derivative** *D at the number x in the domain of f means that*
i) x is a limit point of the domain of f,
ii) if ϵ is a positive number, then there is a positive number δ such that if t is in the domain of f and $|t-x| < \delta$ then

$$|\frac{f(t)-f(x)}{t-x} - D| < \epsilon.$$

Definition 90 *If f is a function which has a derivative at some point, then the* **derivative of f** *is the function, denoted by f', such that for each number x at which f has a derivative, $f'(x)$ is the derivative of f at x.*

Problem 91 *Use either definition of* **derivative** *to show that if $f(x) = x^2 + 1$ then $f'(3) = 6$.*

Problem 92 *Use the definition of* **tangent** *to show that if f is a function whose domain includes $(-1, 1)$ and for each number x in $(-1, 1)$, $-x^2 \leq f(x) \leq x^2$, then the x-axis is tangent to f at the point $(0, 0)$.*

Problem 93 *Use either definition of* **derivative** *to show that if f is a function whose domain includes $(-1, 1)$ and for each number x in $(-1, 1)$, $-x^2 \leq f(x) \leq x^2$, then the derivative of f at the point $(0, 0)$ is 0.*

Theorem 94 *If f is a function, and x is in the domain of f, then f does not have two tangent lines at the point (x,f(x)).*

Definition 95 *If M is a pointset, then the* **closure** *of M is the set consisting of M together with any limit points of M. It is denoted by $Cl(M)$ or by $\bar{M}$.*

Theorem 96 *If M is a point set then $Cl(M)$ is a closed point set.*

Definition 97 *The statement that the point sets H amd K are* **disjoint** *or* **mutually exclusive** *means that they have no point in common.*

Theorem 98 *If f is a function, and x is in the domain of f and f has a derivative at (x,f(x)), then f is continuous at (x,f(x)).*

Theorem 99 *If f is a function whose domain contains $[a, b]$, $x \in (a, b)$, and if $t \in [a, b]$, then $f(x) \geq f(t)$, and f has a derivative at x, then $f'(x) = 0$.*

Definition 100 *If [a,b] is a closed interval, by a* **partition** *of [a,b] is meant a finite increasing sequence $t_0, t_1, \ldots, t_n$ such that $t_0 = a$ and $t_n = b$.*

Notation: Recall that if M is a bounded point set, then either M has a right most point or there is a first point to the right of M. We shall call this number, whichever it is, the **least upper bound of M** and we will denote it by **lub(M)**. Similarly if a set M has a left most point or a first point to the left of M, then we will refer to this point as the **greatest lower bound of M** and denote it by **glb(M)**. Other mathematicians might use the notation, **supremum of M** and **infimum of M** respectively.

Definition 101 *A* **bounded** *function is a function with bounded range.*

For the next 3 definitions, assume that f is a bounded funtion with domain the closed interval $[a, b]$.

Definition 102 *The statement that the number s is a* **Riemann sum** *for f on $[a, b]$ means that there is a partition $t_0, t_1, \ldots, t_n$ of $[a, b]$ and a sequence $x_1, x_2, \ldots, x_n$ of numbers such that*

$$t_0 \le x_1 \le t_1 \le x_2 \le t_2 \le \ldots t_{n-1} \le x_n \le t_n$$

and

$$s = \sum_{i=1}^{i=n} f(x_i)(t_i - t_{i-1})$$

Definition 103 *The statement that the number s is the* **upper Riemann sum** *for f on $[a, b]$ means that there is a partition $t_0, t_1, \ldots, t_n$ of $[a, b]$ and a sequence $y_1, y_2, \ldots, y_n$ of numbers such that for each positive integer i:*

$$y_i = lub\{f(x)|x \in [t_{i-1}, t_i]\}$$

and

$$s = \sum_{1}^{n} y_i(t_i - t_{i-1})$$

Definition 104 *We define a* **lower Riemann sum** *in the same way except that for each positive integer i:*

$$y_i = glb\{f(x)|x \in [t_{i-1}, t_i]\}$$

.

Definition 105 *If F is a bounded function with domain the closed interval [a,b] and P is a partition of [a,b], then $U_P f$ and $L_P f$ denote the upper and lower Riemann sums for f on [a,b] over the partition P, respectively.*

Problem 106 *Let F(x)=0 for each number in [0,1] except 0 and let F(0)=1. Show that if P is a partition of [0,1], then $0 < U_p f$, and if $\epsilon > 0$ then there is a partition P of [0,1] such that $U_P f < \epsilon$.*

Problem 107 *Let F be as in the previous problem. Show that 0 is the only lower Riemann sum for f on [0,1].*

Theorem 108 *If $p_1, p_2, p_3, \ldots$ is a sequence of points in the closed interval $[a, b]$, then there is a point in $[a, b]$ which is not in the sequence $p_1, p_2, p_3, \ldots$.*

Theorem 109 *If p is a limit point of the point set M then there is a sequence of points $p_1, p_2, p_3, \ldots$ of M, all different and none equal to p which converge to p.*

Theorem 110 *If f is a function with domain $[a, b]$, and f is continuous at each number in $[a, b]$, then the range of f is a closed point set.*

Theorem 111 *If $x_1, x_2, x_3, \ldots$ is a sequence of distinct points in the closed interval, $[a, b]$ then the range of the sequence has a limit point.*

Notation: Two often used equivalent statements are:

- If $p_1, p_2, p_2, \ldots$ is a sequence with infinite bounded range then it has a convergent subsequence.
- Every infinite bounded set has a limit point.

Theorem 112 *If f is a function whose domain is $[a, b]$, and f is continuous at each number in $[a, b]$, then the range of f is bounded.*

Definition 113 *If f is a bounded function with domain the closed interval [a,b] then the* **upper integral** *from a to b of f is the greatest lower bound of the set of all upper Riemann sums for f on [a,b] and is denoted by $_U\int_a^b f$. Similarly, the* **lower integral** *from a to b of f is the least upper bound of the set of all lower Riemann sums for f on [a,b] and is denoted by $_L\int_a^b f$.*

Definition 114 *If f is a bounded function with domain [a,b] then the statement that f is* **Riemann integrable** *on [a,b] means that $_L\int_a^b f =_U \int_a^b f$. In this case $_L\int_a^b f$ is called the Riemann integral from a to b of f and is denoted by $\int_a^b f$.*

Theorem 115 *If f is a continuous function whose domain is an closed interval $[a, b]$, then there is a number $x \in [a, b]$ such that if $t \in [a, b]$, then $f(t) \leq f(x)$.*

Theorem 116 *Show that if f is is a function whose domain includes the closed interval [a,b], and for each number x in [a,b], $m \leq f(x) \leq M$, and $P = t_0, t_1, \ldots, t_n$ is any partition of [a,b], then $U_P f \leq M(b-a)$ and $L_P f \geq m(b-a)$.*

Theorem 117 *If f is a bounded function with domain the closed interval [a,b], and P is a partition of [a,b], then $L_P(f) \leq U_P(f)$.*

Theorem 118 *If f is a bounded function with domain [a,b], and for each number x in [a,b], $f(x) \geq 0$, and for some number z in [a,b], $f(z) > 0$ and f is continuous at z, then $_U\int_a^b f > 0$.*

Definition 119 *The statement that the partition Q of the closed interval [a,b] is a* **refinement** *of the partition P of [a,b] means that $P \subseteq Q$.*

Theorem 120 *If f is a bounded function with domain the closed interval [a,b] and P is a partition of [a,b], and Q is a partition of [a,b] and Q is a refinement of P, then $L_P(f) \leq L_Q(f)$, and $U_P(f) \geq U_P(f)$.*

Theorem 121 *If f is a bounded function with domain the closed interval [a,b] then $_L\int_a^b f <_U \int_a^b f$.*

Theorem 122 *If f is a continuous function with domain the closed interval [a,b], and ϵ is a positive number, then there is a partition $x_1, x_2, \ldots, x_n$ of the closed interval [a,b] such that for each positive integer i not larger than n, if u and v are two numbers in the closed interval $[x_{i-1}, x_i]$, then $|f(u) - f(v)| \leq \epsilon$.*

Theorem 123 *If f is a continuous function with domain a closed interval, then the range of f contains only one value or it is a closed interval.*

Theorem 124 *If f is a bounded function with domain the closed interval [a,b] and for each positive number ϵ, there is a partition P of [a,b] such that $U_P(f) - L_P(f) < \epsilon$, then f is Riemann integrable on [a,b].*

Theorem 125 *If f is a continuous function with domain the closed interval [a,b] then f is Riemann integrable on [a,b].*

Theorem 126 *If [a,b] is a closed interval and $c \in (a,b)$ and f is integrable on [a,c] and [c,b] and [a,b], then $\int_a^c f + \int_c^b f = \int_a^b f$. Hint: Show that $\int_a^c f + \int_c^b f$ is not less than $\int_a^b f$, then show it isn't larger.*

Theorem 127 *If f is a continuous function with domain the closed interval [a,b], then there is a number c in [a,b] such that $\int_a^b f = f(c)(b-a)$.*

Theorem 128 *If f is a continuous function with domain the closed interval [a,b] and F is the function such that for each number x in [a,b], $F(x) = \int_a^x f$, then for each number c in [a,b] F has a derivative at c and $F'(c) = f(c)$.*

Theorem 129 *If f is a function with domain the closed interval [a,b], and f has a derivative, $f'(x)$, at each point of [a,b] and f' is continuous at each point in [a,b], then $\int_a^b f' = f(b) - f(a)$.*

Theorem 130 *If f is a function with domain the closed interval [a,b] and there is a sequence $x_1, x_2, \ldots$ of points in [a,b] such that for each positive integer n, $f(x_n) = n$, then there is a point in [a,b] at which f is not continuous.*

Theorem 131 *Suppose m is a nondecreasing function whose domain includes and f has a derivative at each of its points, then there is a number c in (a,b) such that f'(c) is the same as the slope of the line jointing the two points (a,f(a)) and (b,f(b)).*

Chapter 3

Analysis, Neuberger

3.1 Introduction

The following is a list of theorems and definitions which indicate material for a two semester first undergraduate course in analysis. I have given such a course fairly regularly since 1958. During much of the development of this course there was a great deal of discussion with W. S. Mahavier.

Someone who has a good understanding of the material in these notes has a good grounding in calculus and a good start on advanced calculus. However, instructing students in this material has never been a top priority in my classes. The top priority has been to help students gain an ability to do mathematics for themselves - to learn to discover arguments and to solve problems.

I state these theorems and definitions, a few at a time, to the class. They are given some days to try to discover a proof before I call on anyone to ask if they have an argument. From among those who indicate they have an argument I select one to present theirs. If a correct argument is not forthcoming (a very common occurence) I generally ask someone else for an argument. I generally call on students first who I feel are least likely to have a correct argument. I never try to force a presentation from someone who will not claim to have an argument. The course progresses slowly at first if one simply counts the number of theorems proved.

As the course progresses often a remarkable thing happens. Some student who was at first able to finish little or nothing starts to improve their position, sometimes becoming the most able in the class. This sends a powerful message to others. Such an occurence has a profound effect on the whole class and their teacher.

Some months into the class an observer might note that the class has become a group of working intellectuals holding profound discussions on such things as the meaning of some aspects of language. The rate of 'covering' material picks up dramatically.

3.2 Theorem sequence

Definition 1 *The statement that S is a* **segment** *means that there are points a and b so that S is the set of all points between a and b.*

Definition 2 *The statement that I is an* **interval** *means that there are points a and b so that I is the set consisting of a, b and all points between a and b.*

Definition 3 *Suppose M is a point collection. The statement that the point p is a* **limit point** *of M means that every segment containing p contains a point of M different from p.*

Definition 4 *Suppose M is a point collection. The statement that the point p is a* **boundary point** *of M means that every segment containing p contains a point of M and a point not in M.*

Theorem 5 *If a and b are two points, then a is a limit point of the interval $[a, b]$.*

Theorem 6 *Suppose M is a point collection consisting of exactly three points. Then M has no limit point.*

Theorem 7 *If H and K are two segments which have a point in common, then the common part of H and K is a segment.*

Definition 8 *Suppose p is a point and $p_1, p_2, p_3, \ldots$ is a point sequence. The statement that p is a* **sequential limit point** *of $p_1, p_2, p_3, \ldots$ means that if S is a segment containing p, then there is a positive integer N so that p_n is in S for every integer n greater than N*

Theorem 9 *No sequence has two sequential limit points.*

Definition 10 *If M is a point collection, then the statement that M is* **bounded above** *means that there is a point p so that no point of M is to the right of p. The statement that M is* **bounded below** *means that there is a point p so that no point of M is to the left of p. The statement that M is* **bounded** *means that it is bounded above and bounded below.*

Axiom 11 *If M is a point collection which is bounded above, then M has a least upper bound. If M is a point collection which is bounded below, then M has a greatest lower bound.*

Theorem 12 *If $p_1, p_2, p_3, \ldots$ is an increasing sequence which is bounded above, then $p_1, p_2, p_3, \ldots$ has a sequential limit point.*

Definition 13 *The statement that the point collection M is* **closed** *means that if p is a limit point of M, then p is in M.*

Theorem 14 *Suppose M is a point collection which has a limit point and K is the set to which a point belongs if and only if it is a limit point of M. Then K is closed.*

Theorem 15 *Suppose each of H and K is a point collection and p is a limit point of the union of H and K. Then p is a limit point of H or a limit point of K.*

Definition 16 *The statement that the point collection sequence $M_1, M_2, M_3, \ldots$ is* **nested** *means that if n is a positive integer, then M_{n+1} is a subset of M_n.*

Theorem 17 *There is a nested sequence of segments which have no common point.*

Theorem 18 *Every infinite and bounded point collection has a limit point.*

Definition 19 *The statement that the collection G of point collections* **covers** *the point collection M means that if p is in M, then some member of G contains p.*

Theorem 20 *If M is an interval and G is a collection of segments which covers M, then some finite subcollection of G covers M.*

Definition 21 *The statement that the point sequence $p_1, p_2, p_3, \ldots$ is a* **Cauchy sequence** *means that if ϵ is a positive number, then there is a positive integer N so that $|p_n - p_N| < \epsilon$ for every positive integer n greater than N.*

Theorem 22 *Every sequence with a sequential limit is a Cauchy sequence.*

Theorem 23 *If p is a limit point of the point collection M, then there is a sequence $p_1, p_2, p_3, \ldots$ of distinct points of M which has sequential limit point p.*

Theorem 24 *If each of H and K is a closed point collection and H and K have a point in common, then the common part of H and K is closed.*

Theorem 25 *If $I_1, I_2, I_3, \ldots$ is a nested sequence of intervals, then $I_1, I_2, I_3, \ldots$ have a point in common.*

Theorem 26 *If $M_1, M_2, M_3, \ldots$ is a nested sequence of closed and bounded point collections, then $M_1, M_2, M_3, \ldots$ have a point in common.*

Theorem 27 *Every Cauchy sequence has a sequential limit point.*

Definition 28 *The statement that g is a* **simple graph** *means that g is a point collection in the plane so that no vertical line contains two points of g.*

Definition 29 *The statement that the simple graph g is* **continuous** *at the point p of g means that if α and β are two horizontal lines with p between them, then there are two vertical lines h and k with p between them so that every point of g between h and k is also between α and β.*

Theorem 30 *Suppose g is the simple graph consisting of all points (x, x^2) for all numbers x. Then g is continuous at the point $(1, 1)$.*

Theorem 31 *Suppose that g is the simple graph consisting of all points $(x, 1/x)$ for all numbers $x > 0$. Then g is continuous at each of its points.*

Theorem 32 *Suppose g is the simple graph consisting of all points $(x, x^2 + 1/x)$ for all numbers $x \neq 0$. Then g is continuous at each of its points.*

Theorem 33 *Suppose g is an increasing continuous simple graph with domain an interval and L is a horizontal line so that some point of g is above L and some point of g is below L. Then some point of g is on L.*

Theorem 34 *If g is a continuous simple graph with domain an interval then some horizontal line is above g.*

Theorem 35 *If g is a continuous simple graph with domain an interval then the range of g is a point or an interval.*

Theorem 36 *Suppose g is a continuous simple graph with domain an interval and L is a horizontal line so that some point of g is above L and some point of g is below L. Then some point of g is on L.*

Theorem 37 *Suppose g is a continuous simple graph with domain an interval. Then there is a point of g so that no other point of g is above it.*

Theorem 38 *Suppose that $p_1, p_2, p_3, \ldots$ is a bounded sequence. Then some subsequence of this sequence has a sequential limit point.*

Note: From here on the term 'function' is used in place of 'simple graph'.

Definition 39 *Suppose f is a function and c is in the domain of f. The statement that f is* **differentiable** *at c means that*
i) c is a limit point of the domain of f
ii) there is a number d so that if $\epsilon > 0$ there is $\delta > 0$ such that if x is in the domain of f and $0 < |x - c| < \delta$ then

$$|d - (f(x) - f(c))/(x - c)| < \epsilon.$$

Theorem 40 *If f is a function and c is a member of the domain of f at which f is differentiable, then there is only one number d so that (ii) in the above definition holds.*

Theorem 41 *Suppose f is a function and c is a member of the domain of f at which f is differentiable. Then f is continuous at c.*

Theorem 42 *Suppose f is a function whose domain includes the segment (a, b), c is a member of (a, b) at which f is differentiable and $f'(c) > 0$. Then there is a segment S containing c so that*
i) if x is in S and $x < c$, then $f(x) < f(c)$ and
ii) if x is in S and $x > c$, the $f(x) > f(c)$.

Theorem 43 *Suppose f is a function whose domain includes the segment (a, b) and c is a member of (a, b) at which f is differentiable. Suppose also that if x is in (a, b), then $f(x) < f(c)$. Then $f'(c) = 0$.*

Theorem 44 *Suppose $[a, b]$ is an interval and f is a continuous function with domain $[a, b]$ so that $f(a) = 0 = f(b)$ and f is differentiable at each member of (a, b). There is number c in (a, b) so that $f'(c) = 0$.*

Definition 45 *Suppose f is a function and M is a subset of the domain of f. The statement that f is* **uniformly continuous** *on M means that if $\epsilon > 0$ then there is $\delta > 0$ so that if x and y are in M and $|y - x| < \delta$, then $|f(x) - f(y)| < \epsilon$.*

Theorem 46 *Suppose f is a continuous function whose domain includes the interval $[a,b]$. Then f is uniformly continuous on $[a,b]$.*

Definition 47 *Suppose $a < b$ and f is a continuous function whose domain includes $[a,b]$. The statement that U is an* **upper sum** *for f on $[a,b]$ means that there is a positive integer n, an increasing sequence $t_0, t_1, \ldots, t_n$ and a nondecreasing sequence $s_1, s_2, \ldots, s_n$ so that*
i) $t_0 = a, t_n = b$
ii) s_i is in $[t_{i-1}, t_i]$ and $f(s_i) \geq f(x)$ for all x in $[t_{i-1}, t_i], i = 1, 2, \ldots, n$, and
iii) $U = \sum_{i=1}^{n} f(s_i)(t_i - t_{i-1})$.
Lower sums *are defined similarly.*

Definition 48 *Suppose that f is a continuous function whose domain includes the interval $[a,b]$. The statement that f is* **integrable** *from a to b means that there is one and only one number which exceeds no upper sum (for f on $[a,b]$) and is execeeded by no lower sum for f on $[a,b]$.*

Theorem 49 *Suppose f is a nondecreasing function whose domain includes the interval $[a,b]$. Then f is integrable from a to b.*

Theorem 50 *Suppose $a < b$ and f is a continuous function whose domain includes $[a,b]$. Then every lower sum for f on $[a,b]$ is less than or equal to every upper sum for f on $[a,b]$.*

Theorem 51 *If f is a continuous function whose domain includes the interval $[a,b]$, the f is integrable from a to b.*

Theorem 52 *Suppose f is a continuous function whose domain includes the interval $[a,b]$ and c is in $[a,b]$. Then*

$$\int_a^c f + \int_c^b f = \int_a^b f.$$

Theorem 53 *Suppose each of f and g is a continuous function whose domain includes the interval $[a,b]$. Then*

$$\int_a^b f + \int_a^b g = \int_a^b (f+g).$$

Theorem 54 *Suppose f is a continuous function whose domain includes the interval $[a,b]$. There is a number c in $[a,b]$ so that*

$$\int_a^b f = f(c)(b-a).$$

Theorem 55 *Suppose $a < b$, each of f and g is a function whose domain includes the segment (a,b) and each of α and β is a number. If c is in (a,b) and each of f and g is differentiable at c, then $\alpha f + \beta g$ is differentiable at c and*

$$(\alpha f + \beta g)'(c) = \alpha f'(c) + \beta g'(c).$$

Theorem 56 *Suppose f is a continuous function with domain the interval $[a,b]$, c is in $[a,b]$ and g is the function with domain $[a,b]$ so that*

$$g(x) = \int_c^x f \text{ for all } x \text{ in } [a,b].$$

Then $g' = f$.

Theorem 57 *Suppose f is a continuous function whose domain includes the interval $[a,b]$ and F is a function such that $F'(x) = f(x)$ for all x in $[a,b]$. Then*

$$\int_a^b f \; = \; F(b) - F(a).$$

Theorem 58 *Suppose f is a function whose domain includes the interval $[a,b]$ and c is a number. Then*

$$\int_a^b cf \; = \; c\int_a^b f.$$

Definition 59 *The statement that the point collection S in the plane is a* **region** *means that there is a positive number r and a point p in the plane so that S is the collection of all points in the plane which are distant from p by an amount less than r.*

Definition 60 *Suppose M is a point collection in the plane and p is a point in the plane. The statement that p is a* **limit point** *of M means that every region containing p contains a point of M different from p.*

Theorem 61 *Every infinite and bounded point collection in the plane has a limit point.*

Definition 62 *Suppose M is a number collection and each of $f, f_1, f_2, \ldots$ is a funtion whose domain includes M. The statement that $f_1, f_2, \ldots$* **converges uniformly** *to f on M means that if $\epsilon > 0$, there is a positive integer N so that if n is an integer greater than N, then*

$$|f(x) - f_n(x)| < \epsilon \text{ for all } x \text{ in } M.$$

Theorem 63 *Suppose $[a,b]$ is an interval, each of $f, f_1, f_2, \ldots$ is a function with domain $[a,b]$ and $f_1, f_2, \ldots$ converges uniformly to f on $[a,b]$. Then*

$$\int_a^b f_1, \int_a^b f_2, \ldots \text{ converges to } \int_a^b f.$$

Theorem 64 *Suppose $[a,b]$ is an interval, each of $f, f_1, f_2, \ldots$ is a function with domain $[a,b]$ and $f_1, f_2, \ldots$ converges uniformly to f on $[a,b]$. If each of $f_1, f_2, \ldots$ is continuous, then f is continuous.*

Theorem 65 *Suppose each of* $[a, b]$ *and* $[c, d]$ *is an interval and* f *is a continuous function with domain* $[a, b] \times [c, d]$. *Then* f *is uniformly continuous on* $[a, b] \times [c, d]$.

Definition 66 *The statement that the point collection* M *is* **perfect** *means that every point of* M *is a limit point of* M.

Definition 67 *A point collection,* M, *is* **countable** *if there is a sequence,* $p_1, p_2, p_3, \ldots$ *such that for every* m *in* M *there is an integer,* i, *such that* $m = p_i$.

Theorem 68 *No closed and countable number collection is perfect.*

Theorem 69 *Suppose* M *is a closed and bounded point collection in the plane and* G *is a collection of regions covering* M. *Then some finite subcollection of* G *covers* M.

Theorem 70 *Suppose* f *is a continuous function with domain* $[a, b] \times [c, d]$ *and range in* R. *Suppose also that* h *is the function with domain* $[a, b]$ *so that*

$$h(x) = \int_c^d f(x, y)\, dy \text{ for all } x \text{ in } [a, b].$$

Then h *is continuous.*

Theorem 71 *Suppose* f *is a continuous function with domain* $[a, b] \times [c, d]$ *and range in* R. *Then* f *is integrable on* $[a, b] \times [c, d]$.

Theorem 72 *If* f *is a continuous function with domain* $[a, b] \times [c, d]$ *and range in* R, *then*

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_{[a,b] \times [c,d]} f.$$

Theorem 73 *Suppose* f *is a function with domain* $[a, b] \times [c, d]$ *so that each of the partial derivatives* $f_{1,2}$ *and* $f_{2,1}$ *exists and is continuous on* $[a, b] \times [c, d]$. *Then*

$$f_{1,2} = f_{2,1}.$$

Theorem 74 *There is a closed bounded perfect number collection which contains no interval.*

Theorem 75 *Suppose* f *is a continuous function with domain the interval* $[a, b]$. *There is a function* F *so that* $F' = f$.

Theorem 76 *Suppose that each of* u *and* v *is a function with domain* $[a, b] \times [c, d]$ *so that the partial derivatives* u_1, u_2, v_1, v_2 *exist and are continuous on* $[a, b] \times [c, d]$. *If* $u_2 = v_1$ *there is a function* F *on* $[a, b] \times [c, d]$ *so that* $F_1 = u,\ F_2 = v$.

Definition 77 *Suppose $[a, b]$ is an interval. A* **partition** *of $[a, b]$ is a finite ordered sequence, $t_0, t_1, t_2, \ldots t_n$ such that $t_0 = a$, $t_n = b$, and $t_{i-1} < t_i$ for all $i = 1, 2, \ldots n$.*

Definition 78 *Suppose f is a function with domain the interval $[a, b]$. The statement that the graph of f has* **length** *means that there is a number L so that if $t_0, t_1, \ldots, t_n$ is a partition from a to b, then*

$$\sum_{i=1}^{n}[(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1})^2)^{1/2} \leq L.$$

The least such number L is called the length of the graph of f.

Theorem 79 *Suppose f is a function with domain the interval $[a, b]$ and f' is continuous on $[a, b]$. Then the graph of f has length. Moreover, if $g(x) = (1 + f'(x)^2)^{1/2}$ for all x in $[a, b]$, then the length of the graph of f is $\int_a^b g$.*

Theorem 80 *Suppose f is a nondecreasing function on $[a, b]$. There is at most a countable subset of $[a, b]$ on which f is not continuous.*

Theorem 81 *If M is an uncountable set of positive numbers, there is a positive number ϵ so that uncountably many members of M are greater than ϵ.*

Definition 82 *Suppose f is a function and y is in the domain of f. The statement that f has a* **right limit** *at y means that*
i) y is a limit point of the set of all points in the domain of f which are to the right of y, and
ii) there is a number L so that if $\epsilon > 0$, then there is $\delta > 0$ such that if x is in the domain of f and $y < x < y + \delta$, then $|f(x) - L| < \epsilon$. Such a number L is called the right limit of f at y and is denoted by $f(y+)$. Similar statements hold for left limits.

Theorem 83 *Suppose f is a nondecreasing function with domain the segment (a, b). Then if x is in (a, b), f has right and left limits at x.*

Theorem 84 *Suppose f is a function and c is in the domain of f so that f has left and right limits at c and $f(c+) = f(c) = f(c-)$. Then f is continuous at c.*

Theorem 85 *Suppose f is a continuous function whose domain includes the interval $[a, b]$. Then*

$$\left|\int_a^b f\right| \leq \int_a^b |f| \text{ if } a < b.$$

Theorem 86 *Suppose m is a nondecreasing function whose domain includes the interval $[a, b]$. There is a unique number w so that if $t_0, t_1, \ldots, t_n$ is a partition from a to b, then*

$$\sum_{i=1}^{n} t_{i-1}[m(t_i) - m(t_{i-1})] \leq w \leq \sum_{i=1}^{n} t_i[m(t_i) - m(t_{i-1})].$$

Theorem 87 *Suppose h is a function whose domain includes all positive numbers. The following two statements are equivalent:*
i) If $x_1, x_2, \ldots$ is an unbounded increasing sequence of positive numbers, then $h(x_1), h(x_2), \ldots$ converges to L, and
ii) $h(x) \to L$ as $x \to \infty$.

Theorem 88 *Suppose $[a, b]$ is an interval, c is in $[a, b]$ and each of $g_1, g_2, \ldots$ is a continuous function so that*

$$g_n(x) = \int_c^x g_{n-1}, \text{ for all } n = 1, 2, \ldots$$

Then $g_1, g_2, \ldots$ converges uniformly to the zero function on $[a, b]$.

Theorem 89 *If $K > 0$, there is $L > 0$ so that $K^n/n! \leq L2^{-n}$, for all $n = 1, 2, \ldots$.*

Theorem 90 *Suppose that each of f and g is a function and c is a numbers at which both f and g are differentiable. If c is a limit point of the intersection of the domains of f and g, then fg is differentiable at c and*

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

Definition 91 *The statement that the point collection M has* **length** 0 *means that if $\epsilon > 0$ there is a sequence $S_1, S_2, \ldots$ of segments covering M so that $|S_1| + \cdots + |S_n| < \epsilon$ for all positive integers n.*

Theorem 92 *Suppose M is the set of all rational numbers in $[0, 1]$. M has length 0.*

Theorem 93 *Suppose n is a positive integer, $[a, b]$ is an interval and f is a function so that each of $f, f', f'', \ldots, f^{(n+1)}$ is continuous and has domain $[a, b]$. Then*

$$f(x) = \sum_{i=0}^{n} f^{(i)}(a)(x-a)^i/i! + (-1)^n \int_a^x f^{(n+1)}(t)(t-a)^n/n!\, dt$$

for all x in $[a, b]$.

Theorem 94 *Suppose that $a_1, a_2, \ldots$ is a number sequence and r is in $(0, 1)$ so that $|a_{n+1}/a_n| < r$ for all positive integers n. Then $a_1 + a_2 + \cdots$ converges.*

Theorem 95 *Suppose that $a_1, a_2, \cdots$ is a decreasing sequence of positive numbers with sequential limit 0. Then*

$$a_1 - a_2 + a_3 - a_4 + \ldots$$

converges.

Theorem 96 *Suppose that f is a continuous function with domain the interval $[a, b]$ and c is in $[a, b]$. If*

$$f(x) = \int_c^x f$$

for all x in $[a, b]$, then $f(x) = 0$ for all x in $[a, b]$.

Theorem 97 *Suppose c is a number and each of f and g is a function so that g is differentiable at c, f is differentiable at $g(c)$ and c is a limit point of the domain of $f(g)$. Then $f(g)$ is differentiable at c and*

$$(f(g))'(c) = f'(g(c))f'(c).$$

Theorem 98 *Suppose that f is a function whose domain includes the interval $[a, b]$ and c is in $[a, b]$. If the domain of each of $f', f'', \ldots$ includes $[a, b]$ and there is a number M so that $|f^{(n)}(x)| \leq M$ for all x in $[a, b], n = 1, 2, \ldots$, then*

$$f(x) = \sum_{i=0}^{\infty} f^{(i)}(c)(x-c)^i/i!$$

for all x in $[a, b]$.

Theorem 99 *Suppose f is a function whose domain includes the interval $[a, b]$ and c is in (a, b). If the domain of each of each of $f', f'', \ldots$ includes $[a, b]$ and there are positive numbers M and ρ so that*

$$\left| f^{(n)}(x) \right| \leq M\rho^n$$

for all x in $[a, b], n = 1, 2, \ldots$ then there is $\delta > 0$ so that if $|x - c| < \delta$ and x is in $[a, b]$, then

$$f(x) = \sum_{i=0}^{\infty} f^{(i)}(c)(x-c)^i/i!.$$

Theorem 100 *Suppose $a_1, a_2, \ldots$ is a number sequence and $|a_1| + |a_2| + \cdots$ converges. Suppose also that $n_1, n_2, \ldots$ is a sequence of positive integers so that*
i) if i is a positive integer the $i = n_j$ for some positive integer j,
ii) if i and j are two positive integers, then $n_i \neq n_j$.
Then $a_{n_1} + a_{n_2} + \ldots$ converges and equals $a_1 + a_2 + \ldots$.

Chapter 4

Better Mathematics Through Matrices, Tonne

4.1 Introduction

I wrote this book for you, to give you a means to develop your thinking. The subject is rich but simple: just a little algebra is needed. The heart of this book is a series of problems designed to engage your deductive skills and stimulate your ability to find patterns and make generalizations. The goal is to improve your ability to reason; the vehicle is the study of matrices. (Matrices are part of every branch of mathematics and are a frequent tool in, for example, economics, sociology and physics: wherever data is organized.)

As with any exercise, you benefit from it only if you do it yourself. Some problems are challenging. This is by design, to give you an opportunity to expand your abilities. You will find it necessary to dedicate a certain amount of time per week to this endeavor. If you commit some time and effort to this project, this book will help you to think better, to organize your thoughts, and to find mathematical computations a more natural part of everyday life.

I am excited that you have chosen to do this with me.

For review, beginning on page 36, there are some sample problems. This subsection provides a brief summary of the basics of solving equations. If you are confident in solving equations, you may quickly skim that subsection.

At the end of this book there are hints and answers for problems. Last you will find my address in case you want me to look at some of your work and send you my comments.

Work hard and keep an open mind: look for simple ways to do things. If some problem causes difficulty, go on to other problems. Remember to come back to the difficult problem occasionally. Be reluctant to look at the hints and answers until you have done your best on a problem. Then compare your work with the hints and answers. They are there for you to use when you truly need help and also to expand your vision.

Please feel free at any time to make up your own problems and work on them. If this book does not provide enough numerical problems, make up more, solve them, and check your answers. If you think you see some pattern emerging, try to prove it. These activities can be much more rewarding than just doing the problems in order as they appear in the book.

Following is a brief summary of algebra. One of the purposes of this book is to see how a system of matrices compares with the ordinary number system. In particular, we will investigate the properties of matrices corresponding to the properties listed in lines (a) through (j) in the next subsection.

Algebra

2 times 3 is written $2 \cdot 3$.

$$2 \cdot 3 = 3 \cdot 2. \qquad 2 \cdot 4 = 4 \cdot 2. \qquad 2 \cdot 5 = 5 \cdot 2.$$

The power of mathematics is in a letter representing a number. The idea we illustrate above can be stated: if x is a number then $2 \cdot x = x \cdot 2$. More generally, if x and y are numbers then $y \cdot x = x \cdot y$. We can omit the dot and write

$$yx = xy; \quad 2x = x \cdot 2.$$

($x2$ might be confused with x^2 which is $x \cdot x$.)

The rest of this subsection is quite technical. You may just read it lightly.

The fundamental assumptions which we make about numbers are:

1. adding two numbers or multiplying two numbers produces a number;

2. there is a relation on numbers where a number x might be less than a number y; this is written $x < y$ or $y > x$;

3. if each of x, y and z is a number, then

   (a) $x + (y + z) = (x + y) + z$;

   (b) the number zero has the property that $0 + x = x$;

   (c) $-x$ is a number and $x + (-x) = 0$;

   (d) $x + y = y + x$;

   (e) $x(yz) = (xy)z$;

   (f) the number 1 has the property that $1 \cdot x = x$;

   (g) if $x \neq 0$ (x is not zero) then $1/x$ (or $\frac{1}{x}$) is a number and $x \cdot \frac{1}{x} = 1$ (from this it follows that if $xy = 0$ then one of x and y is zero);

   (h) $xy = yx$;

   (i) the number 1 is not the number zero;

   (j) $x(y + z) = xy + xz$;

   (k) if x is not y ($x \neq y$), then either $x < y$ or $y < x$;

   (l) if $x < y$ and $y < z$, then $x < z$;

   (m) if $x < y$, then $x + z < y + z$;

   (n) if $x < y$ and $z > 0$, then $xz < yz$;

   (o) if S is a set of numbers and there is a number which is not less than any member of S, then there is a least number which is not less than any member of S;

   (p) the number 1 is a counting number and if $x < 1$, then x is not a counting number; if y is a counting number, then $y + 1$ is a counting number and, if $y < z$ and $z < y + 1$, then z is not a counting number.

The last two "axioms," (o) and (p), are included for mathematical completeness. They are not necessary for our present study. Also we do very little with the 'less than' relation.

Sample Problems

Problem 1. *Find a number x such that $3x = 6$.*

Here we multiply both sides by $\frac{1}{3}$ and find that

$$x = \tfrac{1}{3} \cdot (3x) = \tfrac{1}{3} \cdot (6) = 2. \qquad x = 2.$$

$$\text{Check: } 3x = 3 \cdot 2 = 6.$$

Problem 2. *Find a number x such that $5x + 4 = 6 - 3x$.*

First we write

$$5x + 3x = 6 - 4; \qquad 8x = 2; \qquad x = \tfrac{2}{8} = \tfrac{1}{4}.$$

$$\text{Check: } 5x + 4 = 5 \cdot (\tfrac{1}{4}) + 4 = \tfrac{5}{4} + \tfrac{16}{4} = \tfrac{21}{4}.$$

$$6 - 3x = 6 - 3 \cdot (\tfrac{1}{4}) = \tfrac{24}{4} - \tfrac{3}{4} = \tfrac{21}{4}.$$

Problem 3. *Find numbers x and y such that $x + y = 6$.*

Here $x = 1$, $y = 5$ is one solution. $x = 2$, $y = 4$ is another. There are many solutions. We may say that the set of all number pairs (x,y) where $y = 6 - x$ is the set of all solutions to the equation $x + y = 6$. Also we may say that the set of all number pairs of the form $(x, 6 - x)$, where x can be any number, is the set of all solutions.

Problem 4. *Find numbers x and y such that $x + y = 6$ and $x - y = 4$.*

$$\left.\begin{array}{rcrl} x+y &=& 6 & \\ x-y &=& 4 & \\ \hline 2x+0 &=& 10 & \text{(Adding the above.)} \end{array}\right\} \quad x = 5; \qquad y = 1.$$

Check: $x + y = 5 + 1 = 6$; $\quad x - y = 5 - 1 = 4$.

It is good practice, and often necessary, to check answers. It is also quite satisfying.

Problem 5. *Find numbers x and y such that $3x + 4y = 7$ and $6x - 5y = 4$.*

Rewrite the first equation; then subtract the second:

$$\left.\begin{array}{rcr} 6x+8y &=& 14 \\ 6x-5y &=& 4 \\ \hline 13y &=& 10 \end{array}\right\} \quad y = \frac{10}{13}.$$

Put y back into one of the original equations and solve for x. Then check your answers by substituting in the original equations.

Alternatively, to find x, we could multiply the first equation by 5 and the second by 4:

$$\left.\begin{array}{lcr} 15x+20y &=& 35 \\ 24x-20y &=& 16 \\ \hline 39x &=& 51 \end{array}\right\} \quad x = \frac{51}{39} = \frac{17}{13}.$$

We verify our answers:

$$3x + 4y = 3 \cdot \frac{17}{13} + 4 \cdot \frac{10}{13} = \frac{51}{13} + \frac{40}{13} = \frac{91}{13} = 7, \quad \text{and}$$

$$6x - 5y = 6 \cdot \frac{17}{13} - 5 \cdot \frac{10}{13} = \frac{102}{13} - \frac{50}{13} = \frac{52}{13} = 4.$$

There is plenty of room for error in this computation. Checking the answers is not only psychologically rewarding but also a practical and logical necessity.

Problem 6. *Find numbers x and y so that both $x + y = 6$ and $x + y = 7$.*

Since 6 is not 7, no solution to this problem is possible.

Problem 7. *Find numbers x and y so that $3x + 4y = 7$ and $9x + 12y = 8$.*

This is also impossible; we realize that the second equation says that $3x + 4y = \frac{8}{3}$, and $\frac{8}{3}$ is not 7.

Problem 8. *Find a number x so that $x^2 = 4$.*

Here 2 and -2 are the answers.

Problem 9. *Find a number x so that $x^2 = -9$.*

Here there is no such number x.

Quadratic Formula

If $ax^2 + bx + c = 0$ then, if $b^2 - 4ac$ is not negative,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We include this formula for your reference, but we will need it seldom in this course.

4.2 Matrices

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a matrix. It is a two-by-two (2×2) matrix. This matrix has two rows (across) and two columns (down). Matrices (the plural of matrix) provide an excellent setting for our endeavor because they are useful and interesting but we need not know very much mathematics to work with them. However, they will provide the practice and challenge we seek.

The product of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ is

$$\begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}.$$

The product of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ is $\begin{bmatrix} a \cdot w + b \cdot y & a \cdot x + b \cdot z \\ c \cdot w + d \cdot y & c \cdot x + d \cdot z \end{bmatrix}$. We write this as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}.$$

Addition of matrices is the way it ought to be:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a + w & b + x \\ c + y & d + z \end{bmatrix}.$$

(To multiply in the same manner would render the subject quite uninteresting.) If we add two matrices, we get a matrix. If we multiply two matrices, we get a matrix.

Problem 1.
(a) Multiply these matrices $\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\right) \begin{bmatrix} 9 & 0 \\ 1 & 2 \end{bmatrix}$.
(b) Multiply these matrices $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \left(\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 1 & 2 \end{bmatrix}\right)$.

Problem 2. *Write down some pairs of matrices and multiply them.*

Problem 3. *Multiply these matrices* $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix}$.

Problem 4. *Cube the matrix* $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. *($M^3 = M \cdot M \cdot M$.)*

Problem 5. *Find, either by guessing or solving equations, a 2×2 matrix* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *such that*

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

That is, find a square root for the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Problem 6. *Is there a formula for the powers (1,2,3,4, ...) of* $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$*?*

Problem 7. *Find, perhaps by guessing, a 2×2 matrix whose cube is the matrix* $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Problem 8. *Is there a formula for the powers (1,2,3,4, ...) of* $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$*?*

Problem 9. *Can you find a matrix whose cube is* $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$*?*

Problem 10. *Can you find a matrix whose cube is* $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$*?*

Problem 11. *Is there a* 2×2 *matrix which multiplies like the number* one*? In other words, find a* 2×2 *matrix* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *such that, for each* 2×2 *matrix* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Problem 12. *Is there a formula for the powers (1,2,3,4, . . .) of* $\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$*?*

Problem 13. *Is there a formula for the powers of* $\begin{bmatrix} 1 & b \\ c & d \end{bmatrix}$ *if* $d = bc$*?*

Problem 14. *Is there a formula for the powers of* $\begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}$*?*

Problem 15. *Find a* 2×2 *matrix* M *such that* $M^2 = M$.

Problem 16. *Find a* 2×2 *matrix* M *such that* $M^2 = -M$.

Problem 17. *Let* M *be the matrix* $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$. *Let* N *be the matrix* $\begin{bmatrix} c & d \\ d & c \end{bmatrix}$. *Show that both* $M + N$ *and* MN *are similar in form to the original matrices.*

Problem 18. *Let* M *be the matrix* $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$. *Let* N *be the matrix* $\begin{bmatrix} c & d \\ 0 & c \end{bmatrix}$. *Show that both* $M + N$ *and* MN *are similar in form to the original matrices.*

Problem 19. *Let* M *be the matrix* $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$. *Let* N *be the matrix* $\begin{bmatrix} e & f \\ 0 & h \end{bmatrix}$. *Show that both* $M + N$ *and* MN *are similar in form to the original matrices.*

If Problem 11 has eluded you so far, try again.

4.3 Solving Equations

Problem 20. *Multiply* $\begin{bmatrix} 1 & 2 \\ 3 & 11 \end{bmatrix}$ *times* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$.

Problem 21. *Find a* 2×2 *matrix* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *such that*

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 11 \end{bmatrix}.$$

Problem 22. *Find a* 2×2 *matrix* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *such that*

$$\begin{bmatrix} 1 & 2 \\ 3 & 11 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Problem 23. *Find a* 2×2 *matrix* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *such that*

$$\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}.$$

Problem 24. *Find a* 2×2 *matrix* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *such that*

$$\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}.$$

Problem 25. *Can you change the matrix after the equal sign in the problem above so that there is a matrix* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *that works?*

Problem 26. *Can you change the first matrix in problem 24 and still have no solution to the equation?*

Problem 27. *By solving equations, find a* 2×2 *matrix* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *such that*

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Problem 28. *By solving equations, find a* 2×2 *matrix* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *whose cube is the matrix* $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Please feel free at any time to make up your own problems and work on them. If I'm not giving you enough numerical problems, make up some more, solve them and check your answers. On the other hand, if you see some pattern emerging, see if you can prove that your pattern is correct. These activities can be much more rewarding than just doing the next problem.

4.4 The Matrix System

In this subsection we ask how much our system of matrices behaves like the number system. We listed the fundamental properties of the number system in the introduction. You might want to look at (a) through (j) on page 35. This section addresses the question: what are the corresponding properties of matrices?

Problem 29. *Is there a* 2×2 *matrix which is like the number zero? In other words, find a* 2×2 *matrix* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *such that, for each* 2×2 *matrix* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The problem above corresponds to property (b) on page 35. Properties (a), (c) and (d) refer only to addition, so they carry directly over to our matrix setting. Thus we turn to those properties which pertain to multiplication.

Problem 30. *(Property f.) Is there a* 2×2 *matrix which is like the number one? In other words, find a* 2×2 *matrix* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *such that, for each* 2×2 *matrix* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Problem 31. *(Property j.) Can you show that if each of* A, B *and* C *is a* 2×2 *matrix, then*

$$A(B + C) = AB + AC?$$

(Here you want to use twelve letters.)

Problem 32. *(Property h.) Can you show that if each of* A *and* B *is a* 2×2 *matrix, then* $AB = BA$*?*

Problem 33. *(Property e.) Can you show that if each of* A, B *and* C *is a* 2×2 *matrix then*

$$A(BC) = (AB)C?$$

Problem 34. *(Property g.) Can you show that if the product of the matrices* A *and* B *is the zero matrix* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, *then either* A *is the zero matrix or* B *is the zero matrix?*

Problem 35. *Is* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *the only answer for problem 30? Call this answer* I. *Then for each* 2×2 *matrix* A,

$$IA = A = AI.$$

Suppose that J also is a 2×2 matrix such that if A is a 2×2 matrix then

$$JA = A = AJ.$$

Can we show that $I = J$?

The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the 2×2 *identity* matrix.

We have seen that matrices behave somewhat like numbers. Matrices fail to have the *commutative* property ($xy = yx$) (see Problem 32). Also in the system of 2×2 matrices there are *divisors of zero*: we can have two non-zero matrices whose product is the zero matrix (see Problem 34).

We take up the notion of reciprocals (property g) in the next section. With matrices it is customary to use the word *inverse* instead of *reciprocal.*

4.5 Inverses

Problem 36. *Find a 2×2 matrix $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ such that*

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem 37. *Find a 2×2 matrix $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ such that*

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem 38. *Find a 2×2 matrix $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ such that*

$$\begin{bmatrix} 1 & 2 \\ 3 & 11 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem 39. *Find a 2×2 matrix $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ such that*

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem 40. *Find a 2×2 matrix $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ such that*

$$\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem 41. *Find a* 2×2 *matrix* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *such that*

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem 42. *Find a* 2×2 *matrix* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *such that*

$$\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem 43. *Find a* 2×2 *matrix* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *such that*

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If you see some pattern emerging, write this down in the form of a problem and see if you can prove that you are right.

In the first three pairs of problems in this section we started with a matrix A and found a matrix B so that $AB = I$ and $BA = I$. Here I represents the *identity* matrix which acts like the number one; that is, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. In this case we say that B is the inverse of A. We could use the term reciprocal, but we wish to remember that we are not in the number system where we can write $\frac{C}{A}$. It is customary to write the inverse of A as A^{-1}. Instead of $\frac{C}{A}$ we must write either CA^{-1} or $A^{-1}C$, because these products might be different.

Let A be the matrix $\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}$. In Problems 40 and 41 we saw that $A^{-1} = \begin{bmatrix} 5/7 & -4/7 \\ -2/7 & 3/7 \end{bmatrix}$. Let K be the matrix $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$. Now Problem 23 reads: Find a matrix X such that $AX = K$. This is like the first sample problem on page 36, isn't it? Multiply both sides (on the left) by A^{-1}:

$$\begin{aligned} A^{-1}AX &= A^{-1}K \\ IX &= A^{-1}K \\ X &= A^{-1}K. \end{aligned}$$

Multiply A^{-1} by K to obtain the answer to Problem 23.

Problem 44. *Suppose that each of* A *and* B *is a* 2×2 *matrix and* $AB = I$. *Is it true that* $BA = I$? *(Unless we know this we must continue to test our alleged inverse on both sides.)*

Problem 45. *Are there matrices which are their own inverses?*

Problem 46. *Find the inverse of* $\begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix}$.

Problem 47. *Find the inverse of* $\begin{bmatrix} 1 & -3 \\ -2 & 11 \end{bmatrix}$.

Problem 48. *Show that if* $d \neq bc$ *then* $\begin{bmatrix} 1 & b \\ c & d \end{bmatrix}$ *has an inverse.*

Problem 49. *Show that* $\begin{bmatrix} 1 & b \\ c & bc \end{bmatrix}$ *has no inverse.*

Problem 50. *Write the inverse of* $\begin{bmatrix} 3 & b \\ c & d \end{bmatrix}$ *when it has one.*

Problem 51. *When does* $\begin{bmatrix} 3 & b \\ c & d \end{bmatrix}$ *fail to have an inverse?*

Problem 52. *Write the inverse of* $\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$ *when it has one.*

Problem 53. *When does* $\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$ *fail to have an inverse?*

Problem 54. *Show that* $\begin{bmatrix} a & b \\ ma & mb \end{bmatrix}$ *has no inverse.*

Problem 55. *Show that* $\begin{bmatrix} a & ma \\ c & mc \end{bmatrix}$ *has no inverse.*

Problem 56. *Show that if* $ad - bc = 0$ *then* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *has no inverse.*

Problem 57. *Write the inverse of* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *when it has one.*

Now you can do problem 44.

4.6 Miscellaneous Problems

I hope that you have enjoyed the first four sections. In this section we investige several different kinds of problems. These problems will motivate the subsequent sections.

These problems might be more thought-provoking than earlier problems. In each problem ask yourself whether you have found all the answers. It is possible that you find some of these problems difficult. You need not quit. Even if you have serious trouble here you can go on and benefit from the subsequent sections. For this section, do what is reasonable for you. Try not to expect either too much or too little from yourself.

In Problems 56 and 57 we saw that the number $ad - bc$ determines when the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse. There is no inverse when $ad - bc = 0$. When $ad - bc \neq 0$ there is an inverse. This number, $ad - bc$, is called the *determinant* of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Problem 58. *What is the determinant of* $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$*? Does this matrix have an inverse?*

Problem 59. *What is the determinant of* $\begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix}$*? Does this matrix have an inverse?*

Problem 60. *What is the product of* $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and* $\begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix}$*? What is the determinant of that product? What do you guess from these calculations?*

Problem 61. *Show that the determinant of* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *is* $(ad-bc)(wz-xy)$.

This says that, for 2×2 matrices, the determinant of the product of two matrices is the product of the individual determinants.

The following problem shows that if the determinant is zero then the matrix fails to have an inverse.

Problem 62. *Suppose that* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *and that* $ad - bc = 0$ *and that* A *has an inverse* A^{-1}*. Find a contradiction.*

Problem 63. *What is the determinant of the matrix in problem 54?*

Problem 64. *What is the determinant of the matrix in problem 55?*

Problem 65. *We have a matrix which acts like the number 1. How many square roots can you find for that matrix? In other words, how many matrices* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *can you find so that*

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}?$$

Problem 66. *Is* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *the only answer for problem 11? Call this answer I. Then for each* 2×2 *matrix* A,

$$IA = A = AI.$$

Suppose that J is a 2×2 matrix such that if A is a 2×2 matrix then

$$JA = A = AJ.$$

Show that $I = J$.

Problem 67. *How many square roots can you find for the zero matrix?*

Problem 68. *How many matrices M can you find where $M^2 = M$? Can you find three?*

Problem 69. *How many matrices M can you find where $M^2 = -M$? Can you find three?*

Problem 70. *Solve equations to find all the answers for problem 67.*

Problem 71. *Solve equations to find all the answers for problem 68.*

Problem 72. *Solve equations to find all the answers for problem 69.*

Problem 73. *Find a square root for the matrix* $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Problem 74. *Find a square root for the matrix* $\begin{bmatrix} 4 & 5 \\ 0 & 9 \end{bmatrix}$.

Problem 75. *Find a square root for the matrix* $\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$.

Problem 76. *Does every 2×2 matrix have a square root?*

Problem 77. *Does every 2×2 matrix have a cube root?*

Problem 78. *Can you multiply these two matrices?*

$$\begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 7 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 6 & 9 & 10 \\ 2 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$$

Problem 79. *Is there a 3×3 matrix which is like the number 1?*

4.7 Commuting Matrices

Problem 80. *How many matrices* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *can you find so that*

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix}?$$

A matrix $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ as above is said to *commute* with the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$; the product one way is the same as the product the other way.

Problem 81. *Find a matrix which does not commute with the matrix* $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Problem 82. *Find all matrices* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *such that*

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix}.$$

That is, find all matrices which commute with $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Let A be the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Let R be a square root of A (that is, R is a 2×2 matrix such that $R^2 = A$). Then

$$RA = R \cdot R^2 = R \cdot R \cdot R = R^2 \cdot R = AR.$$

In summary, $RA = AR$; R *commutes* with A. That is, each square root of A must commute with A. Thus, using the results of the preceding problem, Problem 5 is simplified to the following.

Problem 83. *Find a* 2×2 *matrix* $\begin{bmatrix} w & x \\ 0 & w \end{bmatrix}$ *such that*

$$\begin{bmatrix} w & x \\ 0 & w \end{bmatrix}\begin{bmatrix} w & x \\ 0 & w \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Problem 84. *Use Problem 82 to find a cube root for* $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. *That is, use the fact that the cube root must commute with the original matrix: it must be of the form* $\begin{bmatrix} w & x \\ 0 & w \end{bmatrix}$.

Problem 85. *Find all matrices* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *such that*

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix}.$$

Problem 86. *How many matrices* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *can you find so that*

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

for *every* *matrix* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$*?* (You have to write $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ first. Then I write $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and your matrix has to work.)

Problem 87. *Find all matrices* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *such that*

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

for *every* *matrix* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Problem 88. *How much work is it to find all* 3×3 *matrices which commute with every* 3×3 *matrix?*

Problem 89. *Find all matrices which commute with* $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$.

Problem 90. *Find all matrices which commute with* $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Problem 91. *Show that if* $AB = BA$ *then* $A^3B = BA^3$.

Problem 92. *Find all matrices which commute with* $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$.

Problem 93. *Find all matrices which commute with* $\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$.

Problem 94. *Given a single matrix* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$*, find all matrices which commute with it. Your answer will have the letters* a*,* b*,* c *and* d *in it.*

Problem 95. *Show that the inverse* $\begin{bmatrix} 4/14 & -2/14 \\ -3/14 & 5/14 \end{bmatrix}$ *of the matrix* $\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$ *is of the form* $\begin{bmatrix} w & x \\ \frac{3x}{2} & w - \frac{x}{2} \end{bmatrix}$. (See problem 93.)

Problem 96. *Show that the matrix* $\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$ *itself is of the form* $\begin{bmatrix} w & x \\ \frac{3x}{2} & w - \frac{x}{2} \end{bmatrix}$.

Problem 97. *Show that the square of* $\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$ *is of the form* $\begin{bmatrix} w & x \\ \frac{3x}{2} & w - \frac{x}{2} \end{bmatrix}$.

Problem 98. *Show that* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *is of the form* $\begin{bmatrix} w & x \\ \frac{3x}{2} & w - \frac{x}{2} \end{bmatrix}$.

Problem 99. *Let A be* $\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$. *Show that $A+A^2$ is of the form* $\begin{bmatrix} w & x \\ \frac{3x}{2} & w-\frac{x}{2} \end{bmatrix}$.

Problem 100. *Show that if each of A, B and C is a 2×2 matrix and $AB = BA$ and $AC = CA$ and $D = B + C$ then*

$$AD = DA.$$

Problem 101. *Show that if each of A, B and C is a 2×2 matrix and $AB = BA$ and $AC = CA$ and $D = BC$ then*

$$AD = DA.$$

If p is a number, then by $p\begin{bmatrix} q & r \\ s & t \end{bmatrix}$ we mean $\begin{bmatrix} pq & pr \\ ps & pt \end{bmatrix}$.

Problem 102. *Show that if p is a number then pI commutes with* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Problem 103. *Let A be* $\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$ *as in several earlier problems. Suppose that each of p and q is a number. Show that $pI + qA$ commutes with A.*

Problem 104. *Let A be* $\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$ *as in several earlier problems. Suppose that each of p and q is a number. Show that $pI + qA$ is of the form* $\begin{bmatrix} w & x \\ \frac{3x}{2} & w-\frac{x}{2} \end{bmatrix}$. *(Write w and x in terms of p and q.)*

You have done a hundred problems. Congratulations!

4.8 Matrix Polynomials

Problem 105. *Let A be* $\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$.
(a) Find numbers p and q such that $A^2 = pA + qI$ (I is the identity matrix).
(b) Find numbers p and q such that $A^3 = pA + qI$.
(c) Find numbers p and q such that $A^4 = pA + qI$.

Problem 106. *Let A be* $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.
(a) Find numbers p and q such that $A^2 = pA + qI$.
(b) Find numbers p and q such that $A^3 = pA + qI$.
(c) Find numbers p and q such that $A^4 = pA + qI$.

Problem 107. *Let A be* $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.
(a) Find numbers p and q such that $A^2 = pA + qI$.
(b) Find numbers p and q such that $A^3 = pA + qI$.
(c) Find numbers p and q such that $A^4 = pA + qI$.

Problem 108. *Let A be $\begin{bmatrix} 1 & 2 \\ 3 & 11 \end{bmatrix}$. Guess numbers p and q such that $A^2 = pA + qI$.*

Problem 109. *Let A be $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.*
(a) Guess numbers p and q such that $A^2 = pA + qI$.
(b) Show that your guess is correct.

The preceding problem finds numbers p and q such that $A^2 = pA + qI$ or $A^2 - pA - qI = \mathbf{0}$. (Here $\mathbf{0}$ stands for the zero matrix.) In different terms, we have shown that each 2×2 matrix satisfies a polynomial of degree 2. Here the polynomial is $P(x) = x^2 - px - q$. The word *satisfies* means that the matrix makes the polynomial zero. In problem 108 we see that if A is $\begin{bmatrix} 1 & 2 \\ 3 & 11 \end{bmatrix}$, then A satisfies the polynomial $P(x) = x^2 - 12x + 5$. This means that $P(A) = A^2 - 12A + 5I = \mathbf{0}$.

If you wish, you may wonder about the situation for 3×3 matrices now.

Problem 110. *Let A be $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Problem 106 says that*
$A^2 = 2A - I$,
$A^3 = 3A - 2I$,
$A^4 = 4A - 3I$. *What should A^5 be? What should A^n be?*

Problem 111. *Let A be a matrix such that $A^2 = 2A - I$. Suppose that k is a counting number so that*
$A^k = kA - (k-1)I$. *Show that $A^{k+1} = (k+1)A - kI$.*

Problem 112. *Let A be $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, so that $A^2 = 2A - I$. Is it true that if n is a counting number then $A^n = nA - (n-1)I$?*

Problem 113. *Let A be $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Does the formula obtained above hold if n is 1? If n is 0 (here A^0 should mean the identity matrix I)? If n is -1 (do we get the inverse of A)? If n is $\frac{1}{2}$ (do we get a square root of A)?*

Problem 114. *Let A be $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$. Proceed as in problems 106, 110, 111 and 112 and obtain a formula for the powers of A.*

Problem 115. *Let A be $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$, as above. Does the formula obtained in the last problem hold if n is 1? If n is 0? If n is -1? If n is $\frac{1}{2}$?*

Let A be $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. Can we obtain a formula for the powers of A? I believe there is one, but it seems to be out of my reach.

Problem 116. *Let A be* $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$*, as above. Many matrices (perhaps all the powers of A) can be written in the form* $pA + qI$*. Can* every 2×2 *matrix be written in this form?*

Problem 117. *Let A be a* 2×2 *matrix such that* $A^2 = 2A - 3I$*.*
(a) Show that A has an inverse.
(b) Write a specific matrix A such that $A^2 = 2A - 3I$*.*

Problem 118. *Let A be* $\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$ *as in many earlier problems. Suppose that each of w and x is a number. Show that* $\begin{bmatrix} w & x \\ \frac{3x}{2} & w - \frac{x}{2} \end{bmatrix}$ *is of the form* $pA + qI$*. (Write p and q in terms of w and x.)*

From this last problem and Problem 104, we see that the only matrices which commute with $\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$ are those of the form $pA + qI$, where A is $\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$, itself.

Problem 119. *Let A be* $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$*. According to the method used in ths section, find a formula for the powers of A.*

Problem 120. *Let A be* $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$*. Find numbers p and q such that* $A^2 = pA + qI$*. (I is the* 3×3 *identity matrix. See Problem 79.)*

Problem 121. *Let A be* $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$*. Find numbers p, q and r such that* $A^3 = pA^2 + qA + rI$*.*

Problem 122. *Let A be* $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$*. Find numbers p, q and r such that* $A^3 = pA^2 + qA + rI$*.*

Now, perhaps, it seems that if A is a 3×3 matrix then its third power can be written in terms of the three lesser powers:

$$A^3 = pA^2 + qA + rI.$$

It is not easy to do this using nine letters in A. However, when we did Problem 109, the coefficient q of I was the determinant of A. So it seems that if we did this general problem for 3×3 matrices then we would find the general determinant for 3×3 matrices. It is easy to guess p, I think. You might try this problem now. It sure would make work on the inverses of 3×3 matrices easier if we knew their determinants.

We have done enough, now, to suspect that if A is a 4×4 matrix then its fourth power can be written in terms of the four lesser powers (including A^0, which is the 4×4 identity matrix). You might try this for your favorite 4×4 matrix.

Problem 123. *Let A be a 2×2 matrix such that $A^2 = 4A - 3I$. With the goal in mind of determining numbers p_n and q_n so that, for each counting number n,*

$$A^n = p_n A + q_n I,$$

compute at least six powers of A in terms of A and I. Write out the string of coefficients of A in these and the string of coefficients of I. Find a pattern.

Problem 124. *Use what we have learned to write, in terms of A and I, the square root of a 2×2 matrix A such that $A^2 = 4A - 3I$. Show that your answer works.*

Care to try the cube root? This would be a little more difficult, but obtaining the inverse this way shouldn't be hard.

Problem 125. *Is there a 2×2 matrix M such that M^2 is not the zero matrix but M^3 is the zero matrix?*

Problem 126. *Is there a 3×3 matrix M such that M^2 is not the zero matrix but M^3 is the zero matrix?*

We sure have come a long way together. I hope you are profiting from our journey.

4.9 Complex Numbers

We have not mentioned complex numbers thus far. Indeed we have not allowed them. We said that there is no number whose square is -9.

We might have guessed that $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is a matrix which has no square root. If so, we were frustrated in our attempt to show that there is no square root. There was good reason for this frustration.

Problem 127. *Find a square root for* $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Take either your simplest answer for Problem 127 or else take $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and denote that matrix by the letter i. I have $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Either your choice or mine for i yields $i^2 = -I$.

Complex numbers are usually written $a + bi$. So we will write $aI + bi$ (i as above, or else your choice) and we will have a system of matrices which, we hope, behaves the way complex numbers are supposed to behave. This system consists of all matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (my form). Now you don't have to use your "imagination" to know that there are complex numbers. Our matrices are "real", not "imaginary".

Problem 128. *Let M be $aI + bi$ and N be $cI+di$.*
(a) Show that MN is of the same form.
(b) Show that $MN = NM$.

Problem 129. *Show that if $aI + bi$ is not the zero matrix then it has an inverse.*

Problem 130. *Find a square root for* $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Problem 131. *Find a square root for $aI+bi$. (Here we suppose that $b \neq 0$.)*

Problem 132. *Find all square roots of $-I$.*

Problem 133. *Show that if* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *commutes with* $\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$ *then* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *is a complex number, in the sense that $z = w$ and $x = -y$.*

4.10 Similarity

Problem 134. *Find a matrix* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$, *which has an inverse, such that*

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Problem 135. *Find a matrix* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$, *which has an inverse, such that*

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem 136. *Find a matrix* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$, *which has an inverse, such that*

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

A matrix A is said to be *similar* to a matrix B provided that there is a matrix M, having an inverse, such that

$$AM = MB$$

(alternatively, $B = M^{-1}AM$).

The use of the word *similar* for this purpose is bothersome to some people. We hope that you agree that there is some "similarity" between the given matrices of Problem 134 and also some between the matrices given in Problem 135. Perhaps you will also agree that there is an essential difference between the matrices given in Problem 136. In any event, we will use the word *similar* as in the last paragraph. This is standard in mathematics.

Problem 137. *Show that is A and B are 2×2 matrices which are similar then the determinant of A is the same as the determinant of B.*

Problem 138. *Find a matrix which is similar to* $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Problem 139. *Which matrices are similar to the identity matrix?*

Problem 140. *The matrix* $\begin{bmatrix} 2 & 3 \\ -\frac{5}{3} & -2 \end{bmatrix}$ *is a matrix whose square is $-I$ (Problem 132). Is this matrix similar to* $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$*?*

Problem 141. *Show that if A is* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *and B is* $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ *and $B = M^{-1}AM$ then*

$$p + s = a + d.$$

Problem 142. *Is the matrix* $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ *similar to a matrix of the form* $\begin{bmatrix} p & q \\ r & p \end{bmatrix}$*?*

Problem 143. *Is every matrix* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *similar to a matrix of the form* $\begin{bmatrix} p & q \\ r & p \end{bmatrix}$*?*

Problem 144. *Show that if A is similar to B then A^2 is similar to B^2.*

Problem 145. *Show that if A is similar to B and $R^2 = A$ then B has a square root.*

Problem 146. *Show that if A is similar to B and $R^3 = A$ then B has a cube root.*

Following is the answer for problem 78 which we have placed on this otherwise blank page.

$$\begin{bmatrix} 1\cdot 6+2\cdot 2+5\cdot 1 & 1\cdot 9+2\cdot 1+5\cdot(-1) & 1\cdot 10+2\cdot 2+5\cdot(-1) \\ 3\cdot 6+4\cdot 2+7\cdot 1 & 3\cdot 9+4\cdot 1+7\cdot(-1) & 3\cdot 10+4\cdot 2+7\cdot(-1) \\ 1\cdot 6+2\cdot 2+1\cdot 1 & 1\cdot 9+2\cdot 1+1\cdot(-1) & 1\cdot 10+2\cdot 2+1\cdot(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 6 & 9 \\ 33 & 24 & 31 \\ 11 & 10 & 13 \end{bmatrix}.$$

4.11 Square Roots

We tried to get the square root of $\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$ in Problem 75. Even though we knew that this matrix was the square of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, we found the square root difficult to solve for. In our solution the determinant was crucial.

We have found square roots of the zero matrix (Problem 70). Let us resume our study of square roots by looking at nonzero matrices with determinant zero.

Our goal in this section is to find out exactly which 2×2 matrices have square roots.

The equations for the square root of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are

$$\begin{array}{ll} w^2 + xy = a & (w+z)x = b \\ y(w+z) = c & xy + z^2 = d. \end{array}$$

Problem 147. *Under what condition on the numbers a and c does* $\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$ *have a square root?*

Problem 148. *Under what condition on the numbers b and d does* $\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$ *have a square root?*

Now we wish to discover when $\begin{bmatrix} a & d \\ a & d \end{bmatrix}$ has a square root.

Problem 149. *Does* $\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$ *have a square root?*

Problem 150. *Does* $\begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix}$ *have a square root?*

Problem 151. *Does* $\begin{bmatrix} -2 & 3 \\ -2 & 3 \end{bmatrix}$ *have a square root?*

Can you say what condition on a and d will make $\begin{bmatrix} a & d \\ a & d \end{bmatrix}$ have a square root?

Problem 152. *Show that if* $\begin{bmatrix} a & d \\ a & d \end{bmatrix}$ *has a square root then* $a + d \geq 0$.

Problem 153. *Show that if* $a + d > 0$ *then* $\begin{bmatrix} a & d \\ a & d \end{bmatrix}$ *has a square root.*

Problem 154. *Show that if* $a + mb > 0$ *then* $\begin{bmatrix} a & b \\ ma & mb \end{bmatrix}$ *has a square root.*

Problem 155. *Show that if* $a + d > 0$ *and* $ad - bc = 0$ *then* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *has a square root.*

At this point we might wonder whether $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has a square root whenever $a + d > 0$. Can you think of a problem which we have done which will settle this matter for us?

The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ has a negative determinant, so it has no square root even though $1 + 4 > 0$. Also Problem 127 tells us of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, having a square root, where $a + d = -2$. The determinant is 1.

Problem 156. *Does* $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ *have a square root?*

Problem 157. *When does* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *have a square root?*

The results of the preceeding problem fulfill the mathematical goal I had in writing this book. We know precisely which 2×2 matrices have square roots.

Problem 158. *When does* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *have a fourth root?*

If you have made it this far, you are ready for serious college math. I only wish I could tell you that they will proceed at a reasonable speed.

4.12 Cube Roots

Does every 2×2 matrix have a cube root? We have already asked this question in Problem 77. In case you still would like to work some more on this question we warn you that we give the answer away in the next problem.

Since we did so well with matrices with determinant zero in the last section, we will begin there, and we will use the last section as an outline. Let us see how far we can go toward seeing which 2×2 matrices have cube roots.

The equations for the cube root of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are

$$\begin{aligned} w^3 + 2wxy + xyz &= a & \qquad x(w^2 + xy + wz + z^2) &= b \\ y(w^2 + xy + wz + z^2) &= c & \qquad wxy + 2xyz + z^3 &= d. \end{aligned}$$

Problem 159. *Under what condition on the numbers a and c does* $\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$ *have a cube root?*

Problem 160. *Under what condition on the numbers b and d does* $\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$ *have a cube root?*

Problem 161. *Under what condition on the numbers a, b and m does* $\begin{bmatrix} a & b \\ ma & mb \end{bmatrix}$ *have a cube root?*

Problem 162. *Under what condition on the numbers a, b and d does* $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ *have a cube root?*

Problem 163. *Cube* $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Problem 164. *Show that if* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}^3 = \begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}$ *then* $x = y$ *and* $z = w$.

Problem 165. *Show that if* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}^3 = \begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}$ *then* $(w + x)^3 = 27$.

Problem 166. *Show that if* $\begin{bmatrix} w & x \\ y & z \end{bmatrix}^3 = \begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}$ *then* $4w^3 - 18w^2 + 27w - 14 = 0$ *and* $4x^3 - 18x^2 + 27x - 13 = 0$.

We can see that $w = 2$ and $x = 1$ satisfy the equations above, but this is not really 'solving' the equations. If we were to find the cube root of $\begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}$ honestly, that is, without already knowing the answer from Problem 163, we would have to really solve one of the cubic equations in the problem above. Thus we can see that if the problem were any harder, we would be in serious trouble.

The problem of determining which 2×2 matrices have cube roots can be finished. We have made a start. To finish we would have to do some fancy algebra and know something about cubics or else we would need to devise a different approach to the problem. Similarity is a valuable tool here.

4.13 Inverses For 3×3 Matrices

As we worked through this book we have been working toward independence. I hope that you now feel some confidence in your abilities. As I end my part of your journey, I will give just a little direction for your next few steps.

In attacking the general problem of finding inverses for 3×3 matrices, there are two activities possible. The easier of these is finding classes of 3×3 matrices which have no inverse. In Section 4.5 we found 2×2 matrices which did not have inverses and that can be a guide here.

The second activity is that of finding a general inverse for a 3×3 matrix. I will suggest four ways to attack this problem.

I. The quickest, and perhaps the hardest, way is just to attack the problem head-on and solve the general equations. Once you have done three equations, the other six should not be too bad. We have

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solve for the letters in the second matrix.

II. A second method is to start with easy matrices and find their inverses. In the end we hope to guess a general formula. An intermediate step in this process might be to find the inverse of $\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix}$.

III. Let A be $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. In Section 4.8 we began the problem of finding numbers p, q and r such that $A^3 = pA^2 + qA + rI$. We had a guess for p and thought that r might be the determinant of A. If we finish this problem we will have the determinant of A and this should be a help toward the general inverse. Problem 117 shows how to find A^{-1} if $r \neq 0$.

IV. We might try some systematic guessing in the second matrix so that we have a product like

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}.$$

Then, if we divide the second matrix by k, it should be the inverse of the first matrix.

Knowing how to get the inverse for a 3×3 matrix makes the problem of finding an inverse for a 4×4 identity matrix feasible. We recommend approach IV for 4×4 matrices (we need to study the numerators in our 3×3 inverse).

Thanks for going through this with me. I beleive that you are now ready for the kind of thinking that majoring in mathematics or computers demands.

4.14 Hints

Section 4.2 Multiplication

1. Be neat. Be orderly. Be patient. We multiply inside the parentheses first.

2. Do some. You will learn something.

3. Keep the letters.

4. $M^3 = M \cdot M \cdot M$. Compute

$$\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

5. Guess.

6. Compute several powers.

7. Does the formula for the last problem help?

8. Compute several powers.

9. Does the formula for the last problem help?

10. You can guess this.

11. Guess. Try various numbers for w, x, y and z.

12. Compute several powers.

13. This is like problems 8 and 12.

14. Write some powers.

15. You can guess some.

16. You can guess some.

17. $M + N = \begin{bmatrix} a+c & b+d \\ b+d & a+c \end{bmatrix}.$

18. $M + N = \begin{bmatrix} a+c & b+d \\ 0 & a+c \end{bmatrix}.$

Section 4.3 Solving Equations

20. Keep the letters.

21. $\begin{array}{rr} w+y=1 & x+z=2 \\ y=3 & z=11. \end{array}$

22. Here again we have two equations with w and y and two with x and z:

$$w+2y=1 \quad 3w+11y=0$$

Rewrite the first equation and subtract the second:

$$\left.\begin{array}{rcl} \text{3w+ 6y} & = & 3 \\ \text{3w+11y} & = & 0 \\ \hline \text{-5y} & = & 3 \end{array}\right\} y=-\frac{3}{5}.$$

Put y back into one of the original equations and solve for w. Then check. Treat x and z similarly.

23. Solve equations and check your answer.

24. Solve equations and get a headache.

27. $\begin{array}{rr} w^2+xy=1 & (w+z)x=1 \\ y(w+z)=0 & xy+z^2=1. \end{array}$

28. Note that the second and third equations still factor and that the complicated factor appears in both.

Section 4.4 The Matrix System

29. You guessed it!

30. You can guess this.

31. Be neat. Be orderly. Be patient. Add first and compute

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}+\begin{bmatrix} i & j \\ k & \ell \end{bmatrix}\right).$$

Do the two multiplications, then add:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}+\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} i & j \\ k & \ell \end{bmatrix}.$$

32. How are you doing? If you cannot do this problem, there is a fundamental question: Whose fault is this? (This is a multiple choice question.) Pick A or B:

A) Yours B) The problem's.

33. We hope that this is like problem 31. Use twelve letters again.

34. As in Hint 32 this is a multiple choice question. Do you pick A or B this time?

35. This problem is easy if you look at it right.

Section 4.5 Inverses

36. Solve equations or guess.

37. Solve equations or guess.

38. Solve equations.

39. Guess or solve equations.

40. Guess or solve equations.

41. Guess or solve equations.

42. Does this remind you of Problem 24?

44. This problem is hard.

45. You can guess a couple.

46. Check that the inverse works on both sides.

47. Check both sides.

48. $\begin{bmatrix} 1 & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

$$\begin{array}{rr} w + by = 1 & \quad x + bz = 0 \\ cw + dy = 0 & \quad cx + dz = 1 \end{array}$$

Multiply the first equation by c and subtract the third:

$$\left.\begin{array}{rcl} cx + cby & = & \text{c} \\ cx - dy & = & 0 \\ \hline (cb-d)y & = & \text{c} \end{array}\right\} y = \frac{c}{cb-d};\ y = \frac{-c}{d-bc}.$$

49. We hope that this is no harder than Problem 42.

50. Does Problem 48 help here?

51. The last problem should tell you this.

52. This is easier. A few examples should tell you how to do this problem and the next.

54. $\begin{bmatrix} a & b \\ am & bm \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This produces contradictory equations.

55. Have you been checking your inverses on both sides?

56. Can you survive these equations?

57. Do the earlier inverses in this section tell you the general answer? Alternatively, can you solve the equations, keeping a, b, c and d and using those earlier problems as a guide?

Section 4.6 Problems

58. $1 \cdot 4 - 2 \cdot 3$.

59. $14 - 9$.

60. $\begin{bmatrix} 8 & 17 \\ 18 & 37 \end{bmatrix}$. The answers to the preceding problems were -2 and 5.

61. Compute the determinant of

$$\begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}.$$

62. Let x be the determinant of A. Let y be the determinant of A^{-1}.

65. This is the same as asking that $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ be its own inverse. There are four equations:

$$\begin{array}{rr} w^2 + xy = 1 & (w + z)x = 0 \\ y(w + z) = 0 & xy + z^2 = 1. \end{array}$$

We can do the two cases suggested by problem 61 or else the following two cases:

$$\text{I. } w + z = 0 \quad \text{II. } w + z \neq 0.$$

Get two sheets of paper and do these separately.

66. This problem is about I and J. Which matrix do *you* pick for A in each equation?

67. You can find more than just the zero matrix itself.

68. Can you find ten?

69. Can you find ten?

70. As in problem 65 we have four equations. Which two cases will you consider? Does it help to realize that the determinant of the square root must be zero? $wz - xy = 0$.

71. Here our equations are:

$$w^2 + xy = w \quad (w+z)x = x$$
$$y(w+z) = y \quad xy + z^2 = z.$$

Again $w + z$ appears to be important.

I. $w + z = 1$ II. $w + z \neq 1$.

72. How do your ten answers for Problem 69 compare with those for Problem 68?

73.
$$w^2 + xy = 0 \qquad (w+z)x = 1$$
$$y(w+z) = -1 \qquad xy + z^2 = 0.$$

Where do we start?

74.
$$w^2 + xy = 4 \qquad (w+z)x = 5$$
$$y(w+z) = 0 \qquad xy + z^2 = 9.$$

75. $\begin{bmatrix} w & x \\ y & z \end{bmatrix}^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}.$

$$w^2 + xy = 7 \qquad (w+z)x = 10$$
$$y(w+z) = 15 \qquad xy + z^2 = 22.$$

Can you guess where the matrix $\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$ comes from? It is the square of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. So the equations above have a solution. Now let us forget that there is a solution and try to solve the equations. Can you?

I couldn't either without using the fact that the determinant of the product is the product of the determinants:

$$(wz - xy)^2 = 4.$$

So $wz - xy = 2$ or -2.

76. This problem is interesting because of the matrices which students choose as candidates for having no square root. Which matrices have you chosen? As we have seen, even when there is a square root (as in the previous problem), it can be difficult to find, so, as a practical matter, we need to choose simple matrices.

77. Here the determinant doesn't let us out since each number has a cube root. Does each matrix with determinant zero have a cube root?

78. You can.

79. Yes.

Section 4.7 Commuting Matrices

80. You might have written the zero matrix and the identity matrix. You can write some more.

81. Almost any guess will do.

82. You get four easy (and redundant) equations.

83. $w^2 = 1. \quad 2wx = 1.$

84. $w^3 = 1. \quad 3wx = 1.$

85. $$\begin{array}{ll} w + x = w + y & w + x = x + z \\ y + z = w + y & y + z = x + z. \end{array}$$

86. You know some. Can you produce three?

87. Here you need to be choosy. If a matrix commutes with every matrix, then, in particular, it commutes with the matrices $\cdots$ and $\cdots$.

88. Problems 82 and 85 were enough to do the 2×2 case. How many specific matrices do you think that you will need to do the 3×3 case?

This is more work but no harder that the 2×2 case. (When I ask you about inverses for 3×3 matrices it *will* be a harder problem.)

89. $$\begin{array}{ll} w = w + 2y & 2w + 4x = x + 2z \\ y = 4y & 2y + 4z = 4z. \end{array}$$

90. This matrix is the square of the matrix in problem 82.

91. $A^3B = AAAB = AABA =$

92. $$\begin{array}{rcl} w - 2x & = & w - 3y \\ -3w + 6x & = & x - 3z \\ y - 2z & = & -2w + 6y \\ -3y + 6z & = & -2x + 6z. \end{array}$$

93. Here again you get four equations.

94. We can solve equations, holding on to a, b, c and d.

Or we could look at the answers to the two preceding problems. Then $\begin{bmatrix} w & x \\ \frac{cx}{b} & \end{bmatrix}$ might seem like a good way to begin. The last corner wants to be $w - \frac{(\quad)x}{b}$.

95. $w = \frac{4}{14}. \quad x = \frac{-2}{14}.$

100. $AD = A(B + C) = AB + AC =$

101. $AD =$

102. $\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}.$

103. $(pI + qA)A = pIA + qIAA =$

104. $pI + qA = \begin{bmatrix} p+5q & 2q \\ 3q & p+4q \end{bmatrix} = \begin{bmatrix} w & x \\ \frac{3}{2}x & w - \frac{x}{2} \end{bmatrix}.$

Section 4.8 Matrix Polynomials

105.
(a) $\begin{bmatrix} 31 & 18 \\ 27 & 22 \end{bmatrix} = p\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} + q\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$
(b) $A^3 = A^2A =$
(c) $A^4 = A^3A =$

106.
(a) $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = p\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + q\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$
(b) $A^3 = A^2A =$
(c) $A^4 = A^3A =$

109.
(1) When A is $\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$, $A^2 = 9A - 14I$.
(2) When A is $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $A^2 = 2A - I$.
(3) When A is $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A^2 = 5A + 2I$.
(4) When A is $\begin{bmatrix} 1 & 2 \\ 3 & 11 \end{bmatrix}$, $A^2 = 12A - 5I$.
Use your guess and compute both sides.

Alternatively, you may square $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and solve for p and q.

111. $A^{k+1} = A^kA =$

112. Perhaps there are two aspects to this problem. (1) How certain are you that the formula is correct? (2) Have we done enough to *prove* that the formula is correct?

113. If n is 1,

$$nA - (n-1)I = A - 0I = A = A^1.$$

If n is 0,

$$nA - (n-1)I = 0A - (-1)I = I = A^0.$$

If n is -1,

$$nA - (n-1)I = -A + 2I.$$

If n is $\frac{1}{2}$,

$$nA - (n-1)I = \frac{1}{2}A + \frac{1}{2}I.$$

114. 1, 3, 7, 15, 31. Is there a pattern?
These are coefficients, p, of A in the $pA + qI$ forms of the first five powers of A. Where do the coefficients of I come from?

115. I. Yes. II. Yes. III. If the formula holds where it isn't supposed to hold then

$$A^{-1} = (2^{-1} - 1)A - 2(2^{-2} - 1)I.$$

Does this check?
IV. Let B be $(2^{\frac{1}{2}} - 1)A - 2(2^{\frac{1}{2}-1} - 1)I$. Is B^2 equal to A?

116. Is it likely that any old 2×2 matrix will work?

117. (a) Multiply our given equation on both sides by the alleged A^{-1} and solve for A^{-1}.
(b) The quantity $a + d$ should be 2 and the determinant should be 3.

118. $pA + qI$

$$= \begin{bmatrix} 5p+q & 2p \\ 3p & 4p+q \end{bmatrix} = \begin{bmatrix} w & x \\ \frac{3x}{2} & w - \frac{x}{2} \end{bmatrix}.$$

119. $A^2 = 7A$.

120. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} p & p & p \\ 0 & p & p \\ 0 & 0 & p \end{bmatrix} + \begin{bmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{bmatrix}.$

121. $\begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} =$

$$\begin{bmatrix} p & 2p & 3p \\ 0 & 1p & 2p \\ 0 & 0 & p \end{bmatrix} + \begin{bmatrix} q & q & q \\ 0 & q & q \\ 0 & 0 & q \end{bmatrix} + \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}.$$

122. Be neat. Be orderly. Be patient.

$$A^2 = \begin{bmatrix} 12 & 20 & 24 \\ 22 & 36 & 50 \\ 8 & 12 & 20 \end{bmatrix}.$$

$$A^3 = \begin{bmatrix} 96 & 152 & 224 \\ 180 & 288 & 412 \\ 64 & 104 & 144 \end{bmatrix}.$$

123. 4, 13, 40, 121. Obscure. How about
8, 26, 80, 242?

124. Let B be $A^{\frac{1}{2}}$ according to the formula which works for counting numbers:

$$B = \frac{\sqrt{3}-1}{2}A - \frac{\sqrt{3}-3}{2}I.$$

125. Since $M^3 = 0$, $detM = 0$.

126. Look at Problem 121 and watch the numbers "move out" in A, A^2, A^3.

Section 4.9 Complex Numbers

127. $\begin{array}{cc} w^2 + xy = -1 & (w+z)x = 0 \\ y(w+z) = 0 & xy + z^2 = -1. \end{array}$

128. $M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

129. Find the inverse of $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

130. $\begin{array}{cc} w^2 + xy = 0 & (w+z)x = -1 \\ y(w+z) = 1 & xy + z^2 = 0. \end{array}$
$z^2 = w^2$; $w^2 - z^2 = 0$; $w + z \neq 0$, so $w - z = 0$: $w = z$. Also, $y = -x$, so that $w^2 - x^2 = 0$.

131. The last problem is a guide.

132. The equations are in Hint 127. Note that x cannot be zero.

133. $$\begin{bmatrix} w & x \\ y & z \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix}.$$

Section 4.10 Similarity

134. It's easy to find a matrix which works, but we are required to find one with an inverse. We need just one, so you may guess. However, a general answer is acceptable.

137. Let p be $det\ A$. Let q be $det\ B$. Let r be $det\ M$.

138. Let M be your favorite matrix with an inverse.

139. Not many.

140. We are back to equations. There are several matrices which work for M.

141. Let M be $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ and suppose that $D = det\ M$. This makes M^{-1} equal
$\frac{1}{D}\begin{bmatrix} z & -x \\ -y & w \end{bmatrix}$.

142. There are two ways to attack this.

I. $w + 2y = pw + rx \quad x + 2z = wq + xp$
$3w + 5y = yp + zr \quad 3x + 5z = qy + pz.$

This looks foreboding. But the last problem says that $2p = 6$, and Problem 137 says that $p^2 - qr = -1$, so that $qr = 10$.

II. We try to fix

$$\frac{1}{wz - xy}\begin{bmatrix} z & -x \\ -y & w \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

so that the upper left corner equals the lower right corner.

143. Do we need to do a few more problems like the last one, or are we ready to use the last one as a prototype?

I. If we follow our work above closely, we can try to show that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is similar to $\begin{bmatrix} \frac{a+d}{2} & 1 \\ r & \frac{a+d}{2} \end{bmatrix}$, where $r = [\frac{a-d}{2}]^2 + bc$.

II. $zaw - xcw + zby - xdy$
$= -yax + wcx - ybz + wdz.$

144. $B = M^{-1}AM.$

145. Try $S = M^{-1}RM.$

146. Try $S = M^{-1}RM.$

Section 4.11 Square Roots

147. The determinant $wz - xy$ of the square root is zero. So the first equation becomes $w(w + z) = a$ and the last equation becomes $z(w + z) = 0$.

149. Let p denote $w + z$. Then, since the determinant of the square root is zero, the square root $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ must be $\begin{bmatrix} 2/p & 3/p \\ 2/p & 3/p \end{bmatrix}$.

150. Let p denote $w + z$.

151. Let p denote $w + z$.

152. Let p denote $w + z$.

153. Let p be $\sqrt{a+d}$.

154. Let p be $\sqrt{a+mb}$.

155. This problem, in essence, is to show that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is either of the form $\begin{bmatrix} a & b \\ ma & mb \end{bmatrix}$, in which case you have already done it, or else of the form $\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$ and you can do it just as you did Problem 148.

156. Here the determinant is not zero, so we are back to the equations

$$\begin{array}{cc} w^2 + xy = -1 & (w+z)x = 0 \\ y(w+z) = 0 & xy + z^2 = -2. \end{array}$$

Neither x nor y can be zero.

157. In Problem 76 we saw that $ad - bc$ cannot be negative. So that is one condition for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to have a square root.

Here we no longer know that the determinant of the square root is zero. Let us write $wz - xy$ as D. Then the equation $w^2 + xy = a$ becomes $w^2 + wz - D = a$ or $w(w+z) = D + a$. Let p denote $w + z$.

158. If R is a fourth root then R^2 is a square root. So we know that, if there is a fourth root, then
(1) $ad - bc \geq 0$ and
(2) either (a) $a + d + 2\sqrt{ad - bc} > 0$
or (b) $a = d$ and $b = c = 0$.

It would be nice if each matrix with a square root had a fourth root also. Suppose that $D = \sqrt{ad - bc}$ and $p = \sqrt{a + d + 2D}$. Suppose that $p > 0$ and that

$$M = \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \frac{1}{p} \begin{bmatrix} D + a & b \\ c & D + d \end{bmatrix}.$$

Does M have a square root?

Section 4.12 Cube Roots

159. If $c = 0$ then there is a cube root.
If $c \neq 0$ then $x = 0$.

160. If $b = 0$ then $\begin{bmatrix} 0 & 0 \\ 0 & d^{1/3} \end{bmatrix}$ is a cube root. If $b \neq 0$ then $y = 0$.

161. See problem 9.

162. It seems prudent to make y zero.

164. $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ must commute with $\begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}$.

165. $y = x$. $z = w$. $27 = 13 + 14$.

166. Since $(w + x)^3 = 27$, $w + x = 3$.

4.15 Answers

I have tried to make the answers accurate. I hope you have your own opinion before you look here. If we get the same answer, that should help your confidence. If we get different answers and your answer works, that should help your confidence even more. If I am right and you are not, then please have the humility to resolve to be more persistent and orderly. Everyone makes mistakes.

Section 4.2 Multiplication

1. $\begin{bmatrix} 193 & 44 \\ 437 & 100 \end{bmatrix}$.

2. Write down the things you learned and also the things you guess to be true.

3. $\begin{bmatrix} w + 2y & x + 2z \\ 3w + 4y & 3x + 4z \end{bmatrix}$.

4. $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

5. We come back to this later.

6. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$.

7. Does the formula above help? (We want the 1/3 power. Check your guess by cubing it.)

8. $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}^n = 7^{n-1} \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$.

9. $7^{-2/3} \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$.

10. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

11. Guess again. We come back to this.

12. $10^{n-1} \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$.

13. $(1+d)^{n-1}\begin{bmatrix} 1 & b \\ c & d \end{bmatrix}$. Under what conditions (on d) can you write a formula for the powers of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

14. $\begin{bmatrix} 1^n & b\frac{d^n-1}{d-1} \\ 0 & d^n \end{bmatrix}$. You saw some pattern in the upper right corner. It is not to be expected that you saw this terse way to express that pattern.

15. We will find all of these later.

16. We will find all of these later. Do you have any comment about the answers you already have for these last two problems?

17. $MN = \begin{bmatrix} ac+bd & ad+bc \\ bc+ad & bd+ac \end{bmatrix}$.
(The opposite corners are equal.)

18. $MN = \begin{bmatrix} ac & ad+bc \\ 0 & ac \end{bmatrix}$.

Section 4.3 Solving Equations

20. $\begin{bmatrix} w+2y & x+2z \\ 3w+11y & 3x+11z \end{bmatrix}$.

21. $w=-2$; $x=-9$. Check:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & -9 \\ 3 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 11 \end{bmatrix}.$$

22. $\begin{bmatrix} 11/5 & 9/5 \\ -3/5 & -2/5 \end{bmatrix}$. Check it.

23. $\begin{bmatrix} 13/7 & -39/7 \\ -8/7 & 24/7 \end{bmatrix}$. Check it.

24. $w-3y=3 \qquad -2w+6y=2$.
The first equation says that $-2w+6y=-6$, contradicting the second. There is no solution.

27. The second equation says that $w+z$ is not zero. Thus the third equation tells us that $y=0$. Then $w^2=1$ and $z^2=1$, but w and z have to have the same sign so that $w+z$ is not zero. Check $\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & -1/2 \\ 0 & -1 \end{bmatrix}$ by squaring. Do you really need to check them both?

28. As above, y=0. $w^3=1$. $w=1$. $z^3=1$. $z=1$. $x=1/3$. Cube $\begin{bmatrix} 1 & 1/3 \\ 0 & 1 \end{bmatrix}$ to check your answer.

Section 4.4 The Matrix System

29. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

30. We tell the answer in Problem 35, so see if you can find it before that.

31. The two sides come out equal.

32. I hope you picked B. You can write down two matrices where the product one way differs from the product the other way.

33. The two sides come out equal.

34. I hope you picked B. Then you can write two non-zero matrices whose product is the zero matrix.

35. You have two equations:

$$IA = A = AI \text{ and } JA = A = AJ.$$

What you pick for A in each is critical. We will come back to this in section five.

Section 4.5 Inverses

36. $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

37. $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

38. $\begin{bmatrix} 11/5 & -2/5 \\ -3/5 & 1/5 \end{bmatrix}$.

39. $\begin{bmatrix} 11/5 & -2/5 \\ -3/5 & 1/5 \end{bmatrix}$.

40. $\begin{bmatrix} 5/7 & -4/7 \\ -2/7 & 3/7 \end{bmatrix}$.

41. $\begin{bmatrix} 5/7 & -4/7 \\ -2/7 & 3/7 \end{bmatrix}$.

42. There is no answer.

43. There is no answer. The matrix
$\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$ seems to be fully defective.

44. You will know how to do this when you finish this section.

45. The first problem in the next section is to find all of these.

46.

$$\begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} -7/2 & 3/2 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} -7/2 & 3/2 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

47. $\begin{bmatrix} 11/5 & 3/5 \\ 2/5 & 1/5 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 1 & -3 \\ -2 & 11 \end{bmatrix}$. Can you write the inverse of $\begin{bmatrix} 1 & b \\ c & d \end{bmatrix}$?

48. $\begin{bmatrix} \frac{d}{d-bc} & \frac{-b}{d-bc} \\ \frac{-c}{d-bc} & \frac{1}{d-bc} \end{bmatrix}$.

49. $w + by = 1$, but $cw + bcy = 0$; that is $c(w + by) = 0$. So $c = 0$. But $cx + bcz = 1$.

51. When $d = \frac{bc}{3}$. The demonstration of this is as in problem 49.

52. $\begin{bmatrix} -\frac{d}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix}$.

53. When $b = 0$ or $c = 0$. Do both cases.

54. $aw + by = 1$ (first equation)
$amw + bmy = 0$ (second equation)
$m(aw + by) = 0$; $m = 0$, but
$am + bmz = 1$ (last equation).

55. $\begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} a & ma \\ c & mc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
The equations are contradictory.

56. We will learn more about this later.

57. $\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$.

Section 4.6 Miscellaneous Problems

58. -2 is the determinant; so the matrix has an inverse. The formula in Problem 57 gives the inverse.

59. The determinant is 5, so the matrix has an inverse.

61. Multiply $(ad-bc)(wz-xy)$ out. It should be the same as the determinant of the matrix in the hint.

62. $AA^{-1} = I$. So $xy = 1$, and x cannot be zero.

63. *0.*

64. *0.*

65. I. Suppose that $w = -z$. Then

$$z^2 = 1 - xy; \quad xy = 1 - z^2; \quad y = \frac{1-z^2}{x}.$$

You can square $\begin{bmatrix} -z & x \\ \frac{1-z^2}{x} & z \end{bmatrix}$. The result is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

II. On the other hand, suppose that
$w + z \neq 0$. Then $x = 0$ and $y = 0$. $w^2 = 1$ and $z^2 = 1$. This seems to give four more answers.

66. $IA = A = AI$ for each matrix A. Let A be J. (If it's true for *each*, then I get to pick.) $IJ = J = JI$. $JA = A = AJ$. I'll pick I for A here. Then $JI = I = IJ$. We conclude that $I = J$.

70. I. Suppose that $w + z \neq 0$. Then $x = 0$ and $y = 0$ and $w^2 = 0$ and $z^2 = 0$ and the zero matrix is the only answer along this path.

II. Suppose that $w + z = 0$.
$z = -w$. $w^2 + xy = 0$.

If $x \neq 0$, $y = -\frac{w^2}{x}$ and
$\begin{bmatrix} w & x \\ -\frac{w^2}{x} & -w \end{bmatrix}$ is a matrix whose square is the zero matrix.

If $x = 0$, then $w = 0$ and $\begin{bmatrix} 0 & 0 \\ y & 0 \end{bmatrix}$ is a matrix whose square is the zero matrix.

71. I. Suppose that $w + z = 1$.

If $x \neq 0$, $y = -\frac{w(1-w)}{x}$ and

$\begin{bmatrix} w & x \\ \frac{w(1-w)}{x} & 1-w \end{bmatrix}$ is a matrix which is its own square. Check it!

If $x = 0$, then $w^2 = w$, $w^2 - w = 0$, and w is 0 or 1. Similarly, z is 0 or 1. This gives four answers, two of which are appropriate here and two are appropriate below: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

II. Suppose that $w + z \neq 1$. Then $x = 0$ and $y = 0$ and $w^2 = w$ and $z^2 = z$.

72. If A is a matrix which is its own square ($A^2 = A$) and $B = -A$ then $B^2 = (-A)^2 = A^2 = A = -B$, so the negative of each solution to the previous problem is a solution to this problem. Also the negative of each solution to this problem is a solution to the previous problem.

73. $y = -x$. So $w^2 - x^2 = 0$ and $z^2 - x^2 = 0$. Thus $w^2 - z^2 = 0$ and $(w + z)(w - z) = 0$. $w + z \neq 0$, so $w - z = 0$; $w = z$.

$2wx = 1$. $x = \frac{1}{2w}$.
$w^2 - (\frac{1}{2w})^2 = 0$
$w^4 - \frac{1}{4} = 0$
$w^4 = \frac{1}{4}$
$w^2 = \frac{1}{2}$ ($-\frac{1}{2}$ cannot be a square)
$w = \frac{1}{\sqrt{2}}$ or $w = -\frac{1}{\sqrt{2}}$.
Check one of the answers.

74. $w^2 + xy = 4 \quad (w + z)x = 5$
$y(w + z) = 0 \quad xy + z^2 = 9$.
$w + z \neq 0$ so $y = 0$ and there are four answers.

75. From the middle equations we see that $y = \frac{3x}{2}$. Thus we can rewrite the first and last equations:

$$2w^2 + 3x^2 = 14 \quad \text{and} \quad 3x^2 + 2z^2 = 44.$$

$$w^2 - z^2 = -15 \text{ or } (w + z)(w - z) = -15.$$

Since $(w + z)y = 15$,
$y = -(w - z) = z - w$. All this seems to going somewhere but I cannot finish without some outside help.

If we use determinants to say that $wz - xy = -2$ then $xy = wz + 2$ and $w^2 + wz + 2 = 7$.

$$w^2 + wz = 5$$

$$\begin{aligned} w(w+z) &= 5 \\ (w-z)(w+z) &= -15 \\ \frac{w}{w-z} &= -\frac{5}{15} \end{aligned}$$

$z = 4w$. $w^2 - (4w)^2 = -15$. $w^2 = 1$ and we get two answers.

(Had we chosen $wz - xy = 2$, we would have two other, unexpected, answers. If you work this out—not impossible but quite a challenge— you should be ready for the section concerning square roots of matrices.)

The point of this problem was to show that a square root problem (having an answer) can be quite difficult — impossible, it seems — if we don't get some help from determinants.

76. We hope that you chose $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. The number -1 has no square root, so this matrix, which is like -1, should have no square root. However, it does. We will make this the object of study in another section.

If you chose $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, you were successful in showing that your choice has no square root.

Indeed, if you choose any matrix A whose determinant is negative, then that matrix cannot have a square root R, for if R were a square root of A then the square of the determinant of R must be the negative number which is the determinant of A.

$$0 \leq (det\ R)^2 = det\ A < 0.$$

Does every matrix whose determinant is not negative have a square root?

77. Maybe we can put this off till later.

78. The computation would not fit here. It is on page 55.

79. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Our argument from problem 35 says that this is the only answer for this problem.

You can also write a 4×4 identity matrix.

Section 4.7 Commuting Matrices

80. Did you write the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ itself? The inverse also must commute. You can write even more.

81. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

82. $\begin{array}{ll} w = w + y & w + x = x + z \\ y = y & y + z = z. \end{array}$

$y = 0$ and $w = z$. The matrices which commute with $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are those of the form $\begin{bmatrix} w & x \\ 0 & w \end{bmatrix}$.

83. The square roots of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are $\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$ and its negative.

84. $\begin{bmatrix} 1 & 1/3 \\ 0 & 1 \end{bmatrix}$.

85. $\begin{bmatrix} w & x \\ x & w \end{bmatrix}$.

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is of this form.

86. You probably wrote the zero matrix and the identity matrix. You can write several more.

87. $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ commutes with $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. So it must be of the form $\begin{bmatrix} w & x \\ 0 & w \end{bmatrix}$ and also of the form $\begin{bmatrix} w & x \\ x & w \end{bmatrix}$. Conclusion?

x must be 0. Perhaps you had guessed that $\begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix}$ always works. Now you know that nothing else always works.

88. If we pick $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, we can do it. Be neat!

It is easier if we pick $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. It is surprising that we don't need three matrices. I wonder what the "easiest" pair to pick might be.

More zeros make the equations easier. You can also do the 4x4 case. (Really!) (Right now!)

89. $\begin{bmatrix} w & \frac{2}{3}(z-w) \\ 0 & z \end{bmatrix}$ or
$\begin{bmatrix} w & x \\ 0 & w+\frac{3}{2}x \end{bmatrix}$. Check both forms.

90. $\begin{bmatrix} w & x \\ 0 & w \end{bmatrix}$.

91. $AABA = ABAA = BAAA = BA^3$.

92. $\begin{bmatrix} w & x \\ \frac{2}{3}x & w-\frac{5}{3}x \end{bmatrix}$. There are other ways to write this. Check yours. Check mine.

93. $\begin{bmatrix} w & x \\ \frac{3}{2}x & w-\frac{x}{2} \end{bmatrix}$. It's fun to check these.

94. $z = w - \frac{(a-d)x}{b}$. Check it.

Of course, if $b = 0$ then we need to begin again. Suppose that $b = 0$.

If $a = d$ and $c \neq 0$ then $z = w$ and $x = 0$.

If $a = d$ and $b = c = 0$ then any matrix $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ works.

If $a \neq d$ while $b = 0$ then $x = 0$ and $y = \frac{c(w-z)}{a-d}$.

100. $AB + AC = BA + CA$
$= (B + C)A = DA$.

101. $ABC =$

103. $pIA + qIAA = A(pI) + A(qI)A$
$= A[pI + (qI)A] = A(pI + qA)$.

104. $w = p + 5q$. $x = 2q$.
Check: $\frac{3x}{2} = 3q$.
$w - \frac{x}{2} = p + 5q - q = p + 4q$.

Section 4.8 Matrix Polynomials

105. (a) $A^2 = 9A - 14I$.
(b) $A^3 = (9A - 14I)A = 9A^2 - 14A$
$= 9(9A - 14I) - 14A = 67A - 126I$.
(c) $A^4 = (67A - 126I)A = 67A^2 - 126A$
$= 67(9A - 14I) - 126A = 477A - 938I$.

106. $2A - I,\ 3A - 2I,\ 4A - 3I$.

107. *(a)* $A^2 = 5A + 2I$.
(b) $A^3 = A^2A = (5A + 2I)A = 5A^2 + 2A$
$\quad = 5(5A + 2I) + 2A = 27A + 10I$.
(c) $A^4 = 145A + 54I$.

108. $A^2 = 12A - 5I$.

109. (a) $12 = 1 + 11$. $p = a + d$. q is the negative of the determinant.

(b) $(a + d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} - (ad - bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2$ work out to the same thing.

110. $5A - 4I$. $nA - (n - 1)I$.

111. $A^{k+1} = A^kA = (kA - (k - 1)I)A$
$= kA^2 - (k - 1)A = k(2A - I) - (k - 1)A$
$= 2kA - kI - (k - 1)A = (k + 1)A - kI$.

112. We have seen that this formula is correct for the first few counting numbers. Problem 111 says that once the formula is true for one counting number it is true for the next. Can we be stopped now? NO! The trait is inherited. Each counting number passes it on to its successor. (To justify all this from the axioms in the introduction we need part (p).)

113. Given: $A^2 = 2A - I$.

So, if $n = -1$,

$$(-A + 2I)A = -A^2 + 2I = -(2A - I) + 2A = I$$

and $-A + 2I$ is the inverse of A. Here A and $-A + 2I$ commute, so we have checked both sides.

If $n = \frac{1}{2}$, $(\frac{1}{2}A + \frac{1}{2}I)^2 = \frac{1}{4}(A + I)^2$
$= \frac{1}{4}(A^2 + 2A + I) = \frac{1}{4}(4A) = A$.

114. 1, 3, 7, 15, 31.   2, 4, 8, 16, 32.
The coefficient of I in each line seems to be twice the preceding coefficient of A, which is $2^{n-1} - 1$. A^n should be

$(2^n - 1)A - 2(2^{n-1} - 1)I.$

Suppose that k is a counting number such that $A^k = (2^k - 1)A - 2(2^{k-1} - 1)I$. We wish to show that

$$A^{k+1} = (2^{k+1} - 1)A - 2(2^k - 1)I.$$

Work this out if you can. The exponents are not too bad. Read below if you must.

$$\begin{aligned} A^{k+1} &= A^k A = (2^k - 1)A^2 - 2(2^{k-1} - 1)A \\ &= (2^k - 1)(3A - 2I) - (2^k - 2)A \\ &= (3 \cdot 2^k - 3 - 2^k + 2)A - 2(2^k - 1)I \\ &= (2 \cdot 2^k - 1)A - 2(2^k - 1)I \\ &= (2^{k+1} - 1)A - 2(2^k - 1)I. \end{aligned}$$

Now, since this formula works for the first counting numbers, it always works.

115. $(2^1 - 1)A - 2(2^0 - 1)I = 1A + 0I = A = A^1.$

$(2^0 - 1)A - 2(2^{-1} - 1)I = 0A + 1I = I = A^0.$

Our alleged inverse equals

$$\begin{aligned} &(2^{-1} - 1)A - 2(2^{-2} - 1)I \\ &= (\tfrac{1}{2} - 1)A - 2(\tfrac{1}{4} - 1)I \\ &= (-\tfrac{1}{2})A - 2(-\tfrac{3}{4})I = (-\tfrac{1}{2})A + \tfrac{3}{2}I \\ &= (-\tfrac{1}{2})\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} + \tfrac{3}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Is it really the inverse?

$$\begin{aligned} B^2 &= (2^{\frac{1}{2}} - 1)^2 A^2 \\ &- 2(2^{\frac{1}{2}} - 1) \cdot 2 \cdot (2^{-\frac{1}{2}} - 1)A + 4(2^{-\frac{1}{2}} - 1)^2 I \\ &= (2 - 2 \cdot 2^{\frac{1}{2}} + 1)A^2 - 4(1 - 2^{\frac{1}{2}} - 2^{-\frac{1}{2}} + 1)A \\ &\qquad + 4(2^{-1} - 2 \cdot 2^{-\frac{1}{2}} + 1)I \\ &= (3 - 2^{\frac{3}{2}})A^2 - 4(2 - 2^{\frac{1}{2}} - 2^{-\frac{1}{2}})A + 4(\tfrac{3}{2} - 2^{\frac{1}{2}})I \\ &= (3 - 2^{\frac{3}{2}})(3A - 2I) - 4(2 - 2^{\frac{1}{2}} - 2^{-\frac{1}{2}})A \\ &\qquad + 4(\tfrac{3}{2} - 2^{\frac{1}{2}})I \\ &= (9 - 3 \cdot 2^{\frac{3}{2}} - 8 + 4 \cdot 2^{\frac{1}{2}} + 4 \cdot 2^{-\frac{1}{2}})A \\ &\qquad + (-6 + 2 \cdot 2^{\frac{3}{2}} + 6 - 4 \cdot 2^{\frac{1}{2}})I \\ &= A. \end{aligned}$$

116. I bet that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ doesn't work.

117. (a) $A^2 = 2A - 3I$
$A = A^{-1}A^2 = 2I - 3A^{-1}$

$3A^{-1} = 2I - A$
$A^{-1} = \frac{1}{3}(2I - A)$.
$A^{-1}A = \frac{1}{3}(2A - A^2) = \frac{1}{3}(3I) = I.$
AA^{-1} is the same thing. Our alleged inverse works.

(b) $\begin{bmatrix} -10 & -1 \\ 123 & 12 \end{bmatrix}$. You picked this one, didn't you?

118.
$$\begin{aligned} p &= \tfrac{x}{2} \\ 4p + q &= w - \tfrac{x}{2} \\ 2x + q &= w - \tfrac{x}{2} \\ q &= w - \tfrac{5x}{2}. \end{aligned}$$
Check: $5p + q = \frac{5x}{2} + w - \frac{5x}{2} = w$;
$3p = \frac{3x}{2}$.

119. $A^{k+1} = A^kA = 7^{k-1}AA = 7^{k-1} \cdot A^2$
$= 7^kA$. (See problem 8.)

120. p must be both 2 and 3, so this problem cannot be done: there is no answer.

We found something (problem 109) that worked for 2×2 matrices. What can we do for 3×3 matrices?

121. $3p + q = 6$; $2p + q = 3$.
$p = 3$; $q = -3$.
$p + q + r = 1$; $r = 1$. These check in all equations.

122.
$$\begin{aligned} 96 &= 12p + q + r \\ 180 &= 22p + 3q \\ 64 &= 8p + q \\ 192 &= 24p + 3q \\ 12 &= 2p; \;\; p = 6. \end{aligned}$$
(Had you guessed that p would be 6?)
$q = 16$; $r = 8$. We need to check our values for p, q and r in all nine equations.

123. These are $3^2 - 1$, $3^3 - 1$, $3^4 - 1$, etc. Our guess is that

$$A^n = \frac{3^n - 1}{2}A - \frac{3^n - 3}{2}I.$$

We finish as in problem 114.

124. Compute B^2. After half a dozen lines of careful algebra, you will have A.

125. Let M be $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$. Then according to problem 109, $M^2 = (w+z)M$. Thus $M^3 = (w+z)^2 M$. Since $M^3 = 0$, and M is not the zero matrix (else $M^2 = 0$), $w + z = 0$, but this makes M^2 equal the zero matrix also.

126. The diagonal elements $\begin{bmatrix} p & & \\ & q & \\ & & r \end{bmatrix}$ keep the higher powers from becomming the zero matrix as things "move out". Try replacing these three central numbers with zeros.

Section 4.9 Complex Numbers

127. You may solve the equations in the hint and be done with Problem 132. Otherwise you may make some judicious choices for w, x, y and z to get a specific answer.

128.
$$\begin{aligned} &(a) \quad (aI + bi)(cI + di) \\ &= aI(cI + di) + bi(cI + di) \\ &= acI + adi + bci + bdi^2 \\ &= acI + (ad + bc)i - bdI \\ &= (ac - bd)I + (ad + bc)i. \end{aligned}$$

(b) Since i and I commute, M and N commute, as we can see in computations similar to those above. Perhaps it is more satisfying to multiply $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$ and $\begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

129. Problem 57 says that the answer is $\begin{bmatrix} \frac{a}{a^2+b^2} & \frac{b}{a^2+b^2} \\ \frac{-b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{bmatrix}$. Should we check this? (Note that it is of the form $cI + di$.)

You might just say that $a^2 + b^2$ is the determinant of the matrix and this is not zero unless the matrix is the zero matrix. However, this might be less interesting.

130. If we assume that $w - x = 0$, we run into a dead end. However, if we assume that $w + x = 0$, we find two answers, one of which is $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. Note that this is of the form of the matrices which we are calling complex numbers.

131. Our equations come down to

$$w^2 - \frac{b^2}{4w^2} = a$$

or

$$4(w^2)^2 - 4aw^2 - b^2 = 0.$$

By the quadratic formula we have

$$w^2 = \frac{4a \pm \sqrt{16a^2 + 16b^2}}{2 \cdot 4}.$$

$$w^2 = \frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 + b^2}.$$

Since $b \neq 0$ we must choose

$$w^2 = \frac{a}{2} + \frac{1}{2}\sqrt{a^2 + b^2}$$

lest w^2 be negative. This gives two choices for w. Let us pick

$$w = \sqrt{\frac{a}{2} + \frac{1}{2}\sqrt{a^2 + b^2}}.$$

$$z = w. \quad x = \tfrac{-b}{2w}. \quad y = -x.$$

Our answer is in the complex number form. Can you write our answer and square it? It takes some diligence to come out with $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

132. Answer: $\begin{bmatrix} w & x \\ \frac{-1-w^2}{x} & -w \end{bmatrix}$. You might have picked any one of these to call i at the beginning of this section.

133. $\begin{array}{rcll} 2w + 3x & = & 2w - 3y: & y = -x. \\ -3w + 2x & = & 2x - 3z: & z = w. \end{array}$

Section 4.10 Similarity

134. Any matrix $\begin{bmatrix} w & x \\ x & 0 \end{bmatrix}$ where $x \neq 0$. (If $x \neq 0$, the determinant is not zero).

135. Any matrix $\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}$ where $xy \neq 0$.

136. Here both w and x must be zero so that $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ cannot have an inverse.

137. $pr = rq$ and $r \neq 0$.

138. If your matrix M doesn't commute with $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $M^{-1}AM$ is a different matrix which is similar to $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

139. Only the identity matrix is similar to the identity matrix. $M^{-1}IM = I$.

140.
$$\begin{aligned} 2w + 3y &= x \\ 2x + 3z &= -w \\ -\tfrac{5}{3}w - 2y &= z \\ -\tfrac{5}{3}x - 2z &= -y. \end{aligned}$$
If we pick $w = 1$ and $z = 1$ then M is $\begin{bmatrix} 1 & -2 \\ -\frac{4}{3} & 1 \end{bmatrix}$.

141. Remember that $wz - xy = D$. When we add the first and last entries in $M^{-1}AM$, we get $\frac{1}{D}(zaw - xdy - xya + wdz)$ which is $a + d$. (Right?)

142. I. Since $qr = 10$, we might be tempted to gamble and say $q = 1$ and $r = 10$. Our gamble wins if $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ is similar to $\begin{bmatrix} 3 & 1 \\ 10 & 3 \end{bmatrix}$. You know how to show that this really is so.

II. We need $zw - 3xw + 2zy - 5xy$
$= -xy + 3wx - 2yz + 5wz$, or

$$4yz = 4wz + 6wx + 4xy.$$

(It's hard to get away from guessing in this business.) Let's make w equal to 2.

$$yz = 2z + 3x + 2xy.$$

Now, for convenience, make y equal to 2. This makes x equal to 0 and we may take z to be 1. $wz - xy = 2$. Our arbitrary choices lead us, accidentally, to the same answer as in part I.

143. I. $w = b$, $z = 1$, $x = 0$ and $y = \frac{d-a}{2}$. There are other guesses. It takes some effort to show that this guess works if $b \neq 0$.

If $b = 0$, try $x = 1$, $w = \frac{a-d}{2}$, $y = 0$, and $z = \frac{2c}{a-d}$. This works if $a \neq d$ and $c \neq 0$. If $a = d$ we didn't even need to start the problem.

If $a \neq d$ and $c = 0$, try $x = z = 1$ and $w = \frac{a-d}{2}$ and $y = \frac{d-a}{2}$.

II.
$(a - d)zw + 2byz = (d - a)xy + 2cwx$. If $b \neq 0$, let's make w equal to b again. Try $y = \frac{d-a}{2}$, $x = 0$ and $z = 1$. This works.

If $b = 0$, let us make $y = 0$, $w = 1$, $z = 1$ and $x = \frac{a-d}{2c}$. This works unless $c = 0$, but the case where $b = c = 0$ is not hard.

144. $B^2 = (M^{-1}AM)(M^{-1}AM)$
$= M^{-1}AAM = M^{-1}A^2M.$

145. $S^2 = (M^{-1}RM)(M^{-1}RM)$
$= M^{-1}R^2M = M^{-1}AM = B.$

146. This works just as above.

Section 4.11 Square Roots

147. If we add these equations we get $(w+z)^2 = a$. Thus $a \geq 0$. If $a = 0$ then c must also be zero if our matrix is to have a square root, and we already know about square roots of the zero matrix. If $a > 0$ one square root is $\begin{bmatrix} \sqrt{a} & 0 \\ \frac{c}{\sqrt{a}} & 0 \end{bmatrix}$ or $\frac{1}{\sqrt{a}} \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$.

Note that there is no square root if $a = 0$ and $c \neq 0$.

148. $\frac{1}{\sqrt{d}} \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$. If $d = 0$ and $b \neq 0$ there is no square root.

149.
$$\begin{bmatrix} 2/p & 3/p \\ 2/p & 3/p \end{bmatrix}^2 = \frac{1}{p^2} \begin{bmatrix} 10 & 15 \\ 10 & 15 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}.$$
$p = \pm\sqrt{5}$.

150. $\frac{-2}{p^2} = 2$. There is no square root.

151. Here $p^2 = 1$, and there is a square root.

152. Here p^2 has to be $a + d$, so $a + d \geq 0$.

153. $\begin{bmatrix} a/p & d/p \\ a/p & d/p \end{bmatrix}$. (If $a + d = 0$, there is no square root unless $\begin{bmatrix} a & d \\ a & d \end{bmatrix}$ is the zero matrix.)

154. $\begin{bmatrix} a/p & b/p \\ ma/p & mb/p \end{bmatrix}$. (If $a + mb = 0$, there is no square root unless $\begin{bmatrix} a & b \\ ma & mb \end{bmatrix}$ is the zero matrix.)

156. Since $x \neq 0$ it follows that $z = -w$, but then $-1 = -2$.

157. Our candidate for square root is

$$\frac{1}{p}\begin{bmatrix} D+a & b \\ c & D+d \end{bmatrix}.$$

If we square this, we see that p^2 must be $a+d+2D$ or $a+d+2\sqrt{ad-bc}$.

If $a+d+2\sqrt{ad-bc} > 0$, our candidate wins.

If $a+d+2\sqrt{ad-bc} = 0$ then

$$a+d = w^2+2xy+z^2$$

$$a+d = w^2+2wz-2D+z^2$$

$$0 = a+d+2D = (w+z)^2$$

$$w=-z \qquad b=c=0$$

$$a = w^2+xy = z^2+xy = d$$

Problems 65, 67 and 132 tell us that such matrices have square roots.

In summary, the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has a square root provided that
(1) its determinant is not negative and
(2) either (a) $a+d+2\sqrt{ad-bc} > 0$
or (b) $a=d$ and $b=c=0$.

158. (1) If $p > 0$
then $wz - xy = \frac{(D+a)(D+d)}{p^2} - \frac{bc}{p^2}$
$= \frac{D^2+(a+d)D+ad-bc}{p^2} = \frac{(a+d+2D)D}{p^2} = D \geq 0.$

(2) (a) If $p > 0$, $w+z+2\sqrt{wz-xy} = p+2\sqrt{D} > 0$, and, according to the previous problem, the square root has a square root.

(b) If $a=d$ and $b=c=0$ then we can choose $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ equal to either $\sqrt{a}I$ or else $\sqrt{-a}i$ (if $a<0$). Problems 65 and 130 give square roots for these and thus fourth roots for the original matrix.

Section 4.12 Cube Roots

159. If $c \neq 0$ and $a \neq 0$, then $x=0$, $z=0$, $w=a^{1/3}$ and $y=ca^{-2/3}$.

There is no cube root if $a=0$ and $c \neq 0$.

160. If $b \neq 0$, then $y=0$, $w=0$, $z=d^{1/3}$ and $x=bd^{-2/3}$, unless $d=0$ in which case there is no cube root.

161. The formula for the powers is
$(a+mb)^{n-1}\begin{bmatrix} a & b \\ ma & mb \end{bmatrix}$. If we make n equal to 1/3, we get a cube root, provided that $a+mb \neq 0$. If $a+mb=0$ and $b \neq 0$ then $\begin{bmatrix} a & b \\ ma & mb \end{bmatrix}$ has no cube root, but is similar to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, which has no cube root.

162. $\begin{bmatrix} a^{1/3} & x \\ 0 & d^{1/3} \end{bmatrix}$. x has to be $\frac{b}{a^{2/3}+a^{1/3}d^{1/3}+d^{2/3}}$. One of a and d must be different from zero.

165. If we add $w^3+3wx^2=14$ and
$3xw^2+x^3=13$ we get $(w+x)^3=27$.

166. If we substitute $3-w$ for x in
$w^3+3wx^2=14$ and $3-x$ for w in
$3xw^2+x^3=13$, we obtain the required equations.

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Chapter 5

First Graduate Course in Differential Equations, Neuberger

5.1 Introduction

This material is indicative of John Neuberger's typical first year graduate course in differential equations. It extends H. S. Wall's undergraduate differential equations course. Wall developed an interest in differenetial equations for the occasion of Ernst Hellinger's arrival as a refugee from Germany in the 1930's. Wall wanted to understand some of Hellinger's interests in order to encourage Hellinger's research revival after several years of great difficulty in Germany. The similarity of much of this material with Courant and Hilbert's "Methods of Mathematical Physics" is no co-incidence. Hilbert, Hellinger, and Courant were prominant figures in GÖttingen's great period. Wall was a visitor for a year at GÖttingen. His teacher was Van Vleck, a sudent of Felix Kleins at GÖttingen. I have never handed out such notes as these except after the course had finished. Each course has it's own character and there are rather striking differences in ability levels from year to year. Some of the courses include much more numerics.

5.2 Theorem Sequence

Theorem 1 *If $M > 0$ there is $K > 0$ so that*

$$(M)^n/n! \leq k(1/2)^n \, \forall \, n = 1, 2, \ldots$$

Definition 2 *Suppose each of $f, f_1, f_2, \ldots$ is a function whose domain includes $[a, b]$. The statement that $f_1, f_2, \ldots$* **converges uniformly** *to f on $[a, b]$ means that if $\epsilon > 0 \; \exists \; N \in Z^+$ so that if $n \geq N$ then $|f_n(t) - f(t)| < \epsilon \, \forall \, t \in [a, b]$.*

Theorem 3 *Suppose that $L > 0$ and each of $f_1, f_2, \ldots$ is a continuous function on $\Re$ so that*

$$|f_n(t)| \leq L \left| \int_o^t |f_{n-1}| \right| \; \forall \; n \in Z^+, t \in \Re.$$

If $[a, b]$ is an interval then $f_1, f_2, \ldots$ **converges uniformly** *to 0 on $[a, b]$.*

Definition 4 *Suppose that each of $f_1, f_2, \ldots$ is a function whose domain includes $[a, b]$. The statement that $\{f_i\}_{i=1}^{\infty}$ is* **uniformly Cauchy** *means that if $\epsilon > 0$ then $\exists \; N \ni$ for all $n, m \geq N, |f_n(t) - f_m(t)| < \epsilon$ for all $t \in [a, b]$.*

Theorem 5 *Suppose f_0 is continuous on $\Re$, each of c and b is a number, and each of $f_1, f_2, \ldots$ is a function on $\Re$ so that*

$$f_n(t) = b + \int_c^t f_{n-1} \; \forall \; t \in \Re, \; n \in Z^+.$$

Show that $\{f_i\}_{i=1}^{\infty}$ is uniformly Cauchy on $\Re$.

Theorem 6 *Under the hypothesis of the previous theorem, let* $f(t) = \lim_{n\to\infty} f_n(t)$ *for all* $t \in \Re$ *and show that* $\{f_i\}_{i=1}^{\infty}$ *converges uniformly to* f *on each interval,* $[a, d]$.

Theorem 7 *Under the hypothesis of the previous theorem, there is a unique continuous function* y *so that*

$$y(t) = b\ + \int_c^t y \ \forall\ t \in \Re.$$

Definition 8 *Let* $\mathbf{E}$ *be the unique funtion from* $\Re \to \Re$ *that is its own derivative and is 1 at 0.*

Theorem 9 *Show that:*
i) $E(t) > 0\ \forall\ t \in \Re$.
ii) E *is increasing.*
iii)$E(t) \geq (1 + t)$ *if* $t \geq 0$.
iv) $E(t)E(s) = E(t + s)\ \forall\ s, t\ \in \Re$.
v) If $L = E^{-1}$ *then* $L(u) + L(v) = L(uv)\ \forall\ u, v > 0$.
vi) $L'(t) = 1/t\ \forall\ t > 0$.

Theorem 10 *Suppose* $a < c < b$, $q \in \Re$, *and each of* p *and* g *is a continuous function on* $[a, b]$. *Suppose also that each of* $y_0, y_1, \ldots$ *is a continuous function on* $[a, b]$ *so that*

$$y_n(t) = q + \int_c^t g + \int_c^t p y_{n-1}\ \forall\ t \in [a, b].$$

There is a function y *on* $[a, b]$ *to which* $y_1, y_2, \ldots$ *converges uniformly on* $[a, b]$. *Moreover,* y *is the unique continuous function* f *on* $[a, b]$ *so that* $f(t) = q + \int_c^t g + \int_c^t pf\ \forall\ t \in [a, b]$.

Definition 11 *An* **integrating factor** *for the ordinary differential equation,* $y' = py + g$, *is a function,* μ, *which is never zero such that if* μ *is multiplied by the differential equation, then the left hand side of the equation can be written as the derivative of* $\mu y'$.

Problem 12 *Find an integrating factor for* $y(c) = q, y' = py + g$.

Problem 13 *Show that if* g *is a continuous function on* $[0, 1]$, *then:*
(i) there is a unique function y *on* $[0, 1]$ *so that* $y(0) = 0 = y(1)$ *and* $-y'' = g$.
(ii) Find as nice a form for your answer as possible.

Note: Study of functional analysis was born with Problem 9!

Definition 14 *A* **metric space** *S is a collection of points with a distance (metric), d, satisfying:*
i) $d(x, x) = 0$
ii) $d(x, y) > 0$ *if* $x \neq y$
iii) $d(x, y) = d(y, x)$
iv) $d(x, y) + d(y, z) \geq d(x, z)$

Definition 15 *A* **complete** *metric space is a metric space in which every Cauchy sequence has a sequential limit (in the space).*

Definition 16 *Let S be a metric space. A function, $f : S \to S$, is a* **contraction mapping** *if $\exists\ 0 < \lambda < 1$ such that $d(f(x), f(y)) \leq \lambda d(x, y)\ \forall\ x, y \in S$.*

Theorem 17 *If S is a complete metric space, $f : S \to S$ is a contraction mapping on S, $x_0 \in S$, and $x_n = f(x_{n-1})\ \forall\ n = 1, 2, \ldots$ then $x_1, x_2, \ldots$ converges to the unique fixed point of f.*

Problem 18 *Find all non-trivial functions y on $\Re$ so that*

$$y(0) = 0 \text{ and } y'(t) = ((y(t)^2)^{1/3}\ \forall\ t \in \Re.$$

Theorem 19 *Suppose that g is a continuous function from $\Re$ to $\Re$ so that $g(t) + g(s) = g(t + s)\ \forall\ t, s$ in $\Re$. Then $\exists\ c \in \Re$ so that $g(t) = ct\ \forall\ t \in \Re$.*

or

Theorem 20 *Suppose f is continuous from $\Re$ to $\Re$ so that $f(t)f(s) = f(t + s)\ \forall\ t, s$. Then either $f = 0$ or $\exists$ a number c so that $f(t) = e^{ct}\ \forall\ t \in \Re$.*

Theorem 21 *Suppose $a < b, a \leq c \leq b$, each of $q_{11}, q_{12}, q_{21}, q_{22}$, f, and g is a continuous function on $[a, b]$, and each of r and s is a number. There is a unique pair u, v each of which is a function on $[a, b]$ so that:*
i) $u(c) = r$, $u' = f + q_{11}u + q_{12}v$
ii) $v(c) = s$, $v' = g + q_{21}u + q_{22}v$.

Definition 22 *A* **linear (vector) space,** *$(S, +, \cdot)$, is a set of points, S, together with two functions: $+ : S \times S \to S$ and $\cdot : \Re \times S \to S$ such that given $x, y \in S$ and $a, b \in \Re$:*
i) $x + y = y + x$
ii) $(x + y) + z = x + (y + z)$
iii) $\exists\ 0 \in S$ *such that* $x + 0 = x$ *all* x
iv)$(a + b)x = ax + bx$
v) $a(x + y) = ax + ay$
vi) If $ax = 0$ then $a = 0$ or x is the zero element of the set.

Definition 23 *A* **linear transformation,** T, *is a function with domain and range a linear space such that given* x, y *in the space and* $\alpha \in \Re$*:*

$$Tx + Ty = T(x+y) \text{ and } \alpha Tx = T(\alpha x).$$

Definition 24 *Suppose* $H = (S, +, \cdot)$ *is a linear space. A norm for* H *is a function* $\|\cdot\|$ *such that if each of* $x, y \in S$ *and* $\alpha \in \Re$ *then:*
i) $\|x\| > 0$ *unless* x *is the zero element in which case* $\|x\| = 0$
ii) $\|ax\| = |a|\|x\|$
iii) $\|x+y\| \le \|x\| + \|y\|$

Definition 25 *A linear transformation,* T, *is* **bounded** *provided* $\exists$ *a number* $M \ni \|Tx\| \le M\|x\|$ *for all* x *in the domain of* T*. The smallest such number* M *is denoted* $|T|$.

Definition 26 $L(\Re^2, \Re^2)$ *is the set of linear transformations from* $\Re^2$ *into* $\Re^2$.

Theorem 27 *If* $a < b$, $a \le c \le b$, $Q : [a,b] \to L(\Re^2, \Re^2)$ *is continuous,* $G : [a,b] \to \Re^2$ *is continuous, and* $\alpha \in \Re^2$, *then there is a unique function* Y *from* $[a,b] \to \Re^2$ *such that:*

$$Y(c) = \alpha \text{ and } Y'(t) = G(t) + Q(t)Y(t) \ \forall \ t \in [a,b].$$

Definition 28 *If* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L(\Re^2, \Re^2)$ *then* $|\begin{pmatrix} a & b \\ c & d \end{pmatrix}|$ *is the least number* M *such that*

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \le M \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| \ \forall \ \begin{pmatrix} x \\ y \end{pmatrix} \in \Re^2.$$

Note: The greatest lower bound of all such M is *in* the set of all such M.

Problem 29 *Find all functions* f *on* $[0,1]$ *and all* $\lambda \ne 0$ *so that* $f'' = -(1/\lambda)f$ *and* $f(0) = 0 = f(1)$.

Note: These eigenfunctions form a basis for the space of continuous functions on $[0,1]$ with root mean square norm: $\|f\| = \sqrt{\int_0^1 f^2}$.

Theorem 30 *Suppose that each of* H *and* K *is a normed linear space and* A *is a linear transformation from* H *to* K*. Then* A *is bounded if and only if* A *is continuous.*

Theorem 31 *Suppose* $a < b,\ a \le c \le b,\ Q$ *is a continuous function from* $[a,b] \to L(\Re^2,\Re^2),\ G$ *is a continuous function from* $[a,b] \to L(\Re^2,\Re^2)$, *and* $\alpha \in L(\Re^2,\Re^2)$. *Then* $\exists$ *a unique function* Y *from* $[a,b] \to L(\Re^2,\Re^2)$ *so that*

$$Y(c) = \alpha \text{ and } Y'(t) = G(t) + Q(t)Y(t)\ \forall\ t \in [a,b].$$

Note: Q(t)Y(t) means the composition of the two linear transformations.

Definition 32 *Suppose* $a < b$ *and* Q *is a continuous function from* $[a,b]$ *to* $L(\Re^2,\Re^2)$. *Denote by* M *the function from* $[a,b] \times [a,b] \to L(\Re^2,\Re^2)$ *so that if each of* s *and* r *is in* $[a,b]$, *then*

$$M(r,s) = Y(r) \text{ where } Y(s) = I \text{ and } Y'(t) = Q(t)Y(t)\ \forall\ t \in [a,b].$$

Note: M depends on $[a,b]$ and on Q. M is somewhat like E.

Theorem 33 *Suppose* $a < b,\ Q$ *is a continuous function from* $[a,b]$ *to* $L(\Re^2,\Re^2)$ *and* M *is as in the previous defintion. Then*

$$M(r,s)M(s,q) = M(r,q)\ \forall\ r,s,q \in [a,b].$$

Note: $M(r,s)M(s,r) = M(r,r) = I$. $M(r,s)$ and $M(s,r)$ are inverses. e^x and e^{-x} are reciprocals. M never has determinant zero and e^x is never zero.

Problem 34 *Suppose that* $Q(t)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\ \forall\ t \in \Re,\ \begin{pmatrix} x \\ y \end{pmatrix}$ *in* $\Re^2$. *Find an expression for* $M(t,s)\ \forall\ t,s \in \Re$.

Theorem 35 *Under the hypothesis of Thereom 33 and using Definition 32, show that* $M_2(s,r) = -M(s,r)Q(r)\ \forall\ r,s \in [a,b]$.

Hint: Need to show derivative has to exist. Establish product rule first.

Theorem 36 *Suppose* $a \le c \le b,\ Q$ *is a continuous function from* $[a,b] \to L(\Re^2,\Re^2),\ G$ *is a continuous function from* $[a,b] \to \Re^2$, *and* Y *is a function from* $[a,b]$ *to* $\Re^2$ *so that* $Y'(t) = G(t) + Q(t)Y(t)\ \forall\ t \in [a,b]$. *Then*

$$Y(t) = M(t,c)Y(c) + \int_c^t M(t,s)G(s)ds\ \forall\ t \in [a,b].$$

Hint: Use an integrating factor for $Y' = G + QY$ to obtain the result.

Problem 37 *Find* $A_1, A_2 \in L(\Re^2,\Re^2), G : [0,1] \to \Re^2$, *and* $Q : [0,1] \to L(\Re^2,\Re^2)$ *so that the problem of finding* $Y : [0,1] \to \Re^2$ *such that*

$$A_1 Y(0) + A_2 Y(1) = 0 \text{ and } Y' = G + QY.$$

is equivalent to finding $y : [0,1] \to \Re$ so that

$$y(0) = 0 = y(1) \text{ and } -y'' = g.$$

Theorem 38 *Suppose $A \in L(\Re^2, \Re^2)$ and $Q(t) = A \ \forall \ t \in \Re$. Then M, as defined in Definition 32, has the property that*

$$M(t,s) = M(t-s,0) \ \forall \ t,s \ \in \Re.$$

Moreover, if $T(r) = M(r,0) \ \forall \ r \in \Re$, then

$$T(0) = I,$$

$$T(t)T(s) = T(t+s) \ \forall \ t,s \in \Re \text{ and}$$
$$T(t) = e^{tA} \ \forall \ t \in \Re.$$

Theorem 39 *Suppose that $(a,b) \in \Re^2$, each of α and β is a positive number, and f is a continuous function from $[a-\alpha, a+\alpha] \times [b-\beta, b+\beta] \to \Re$. Suppose moreover that $\exists \ K > 0$ so that the following* **Lipschitz** *condition holds:*

$$|f(t,x) - f(t,y)| \leq K|x-y| \ \forall \ t \in [a-\alpha, a+\alpha], \ x,y \in [b-\beta, b+\beta].$$

Then there is $\gamma > 0$ for which there is a unique function $y : [a-\gamma, a+\gamma] \to \Re$ so that

$$y(a) = b \text{ and } y'(t) = f(t, y(t)) \ \forall \ t \in [a-\gamma, a+\gamma].$$

Note: The previous theorem is called a local existence theorem and the argument given generalizes to n dimensions.

Note also: The previous theorem might well be stated in the following way.

Theorem 40 *Let $\Omega = [a-\alpha, a+\alpha] \times [b-\beta, b+\beta]$, f be a continuous function with bound, B, on Ω and for each $t \in [a-\alpha, a+\alpha]$ assume that f is Lipschitz with Lipschitz constant, K, on $[b-\beta, b+\beta]$. If $0 < \gamma < min\{1/K, \beta/K, \alpha\}$ then there exists a unique solution to*

$$y(a) = b \text{ and } y' = f(t,y) \ \forall \ t \in [a-\gamma, a+\gamma].$$

Theorem 41 *Suppose $a < b, Q : [a,b] \to L(\Re^2, \Re^2)$ is continuous, and $A_1, A_2 \in L(\Re^2, \Re^2)$. Let M and Q be defined from Definition 32. TFAE:*
i) If $G : [a,b] \to \Re^2$ is continuous, then $\exists$ unique $Y : [a,b] \to \Re^2$ so that $A_1Y(a) + A_2Y(b) = 0$, and $Y' = G + QY$.
ii) $[A_1 + A_2M(b,a)]^{-1}$ exists.

Problem 42 *Restate Problem 13 in vector and matrix notation by defining A_1, A_2, Q, G, and Y so that i) and ii) are true and imply the solution to 13.*

Theorem 43 *Suppose* $A_1, A_2 \in L(\Re^2, \Re^2), Q : [a,b] \to \Re$ *is continuous, and* $(A_1 + A_2 M_Q(b,a))^{-1}$ *exists. Then there exists a function* $K : [a,b] \times [a,b] \to L(\Re^2, \Re^2)$ *so that if* $G : [a,b] \to \Re^2$*, is continuous, then the unique function* $Y : [a,b] \to \Re^2$ *satisfying* $A_1 Y(a) + A_2 Y(b) = 0$ *and* $Y' = G + QY$ *is given by*

$$Y(t) = \int_a^b K(t,s)G(s)ds \ \forall \ t \in [a,b].$$

Theorem 44 *If each of* $g, q : [a,b] \to \Re$ *is continuous, and* $A_1, A_2 \in L(\Re^2, \Re^2)$*, then there exists a unique function* $y : [a,b] \to \Re$ *satisfying*

$$y'' - qy = g \quad and \ A_1 \begin{pmatrix} y(a) \\ y'(a) \end{pmatrix} + A_2 \begin{pmatrix} y(b) \\ y'(b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (*)$$

if and only if $(A_1 + M(b,a)A_2)^{-1}$ *exists where* $Q(t) = \begin{pmatrix} 0 & 1 \\ q(t) & 0 \end{pmatrix}, t \in [0,1]$*, and* $M = M_Q$*. Furthermore, y may be wrtten as*

$$y(t) = \int_a^b k(t,s)g(s)ds\$\forall \ t \in [a,b]$$

for some continuous function k on $[a,b] \times [a,b]$.

Theorem 45 *Suppose* $Q : [a,b] \to L(\Re^2, \Re^2)$ *is continuous. Then*

$$det(M_Q(t,s)) = e^{\int_s^t trQ} \ \forall \ t, s \in [a,b].$$

Theorem 46 *Under the hypothesis of Theorem 44,*

$$k(t,s) = k(s,t) \ \forall \ s,t \in [a,b] \iff detA_1 = detA_2.$$

Note: $k =$ one of small corners of K from Theorem 43.

Problem 47 *Find an inequality for* $\|A_n \cdots A_1 x - B_n \cdots B_1 x\| \leq \ldots$

Note: The inequality is key to a numerical process and one gets a good error estimate with a good inequality.

Problem 48 *Write a code to solve* $Y' = G + QY$ *for* $Y \in \Re^2$.

Definition 49 *Suppose* $f_1, f_2, \ldots$ *is a sequence of functions from* $[a,b] \to \Re$*. The statement that the sequence is* **equi-continuous** *at c means:* $\forall \ \epsilon > 0 \ \exists \ \delta > 0$ *such that if* $x \in [a,b]$*, and* $|x - c| < \delta$*, then* $|f_n(x) - f_n(c)| < \epsilon \ \forall \ n \in Z^+$.

Definition 50 *Suppose $f_1, f_2, \ldots$ is a sequence of functions from $[a, b] \to \Re$. The statement that the sequence is* **uniformly equi-continuous** *on $[a, b]$ means: $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that if $x, c \in [a, b]$ such that $|x - c| < \delta$ then $|f_n(x) - f_n(c)| < \epsilon\ \forall\ n \in Z^+$.*

Definition 51 *The sequence $f_1, f_2, \ldots : [a, b] \to \Re$ is* **uniformly bounded on [a,b]** *if $\exists\ M \in \Re \ni f_n(x) \leq M\ \forall\ x \in [a, b]$.*

Theorem 52 *Pointwise equi-continuous on I implies uniformly equi-continuous on I.*

Theorem 53 *If $f_1, f_2, \ldots : [a, b] \to \Re$ is a uniformly bounded equi-continuous sequence of functions then $\{f_i\}_{i=1}^{\infty}$ has a uniformly convergent subsequence.*

Theorem 54 *Suppose $a < b$, k is a continuous real valued function on $[a, b] \times [a, b]$ (like k in Theorem 44). Suppose also that each of $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ is a sequence of continuous functions on $[a, b]$ so that $\{g_i\}_1^{\infty}$ is uniformly bounded and*

$$f_n(t) = \int_a^b k(t, s) g_n(s) ds\ \forall\ t \in [a, b],\ n \in Z^+$$

Then $\{f_i\}_{i=1}^{\infty}$ is uniformly bounded and equi-continuous.

Claim: Previous two theorems have to do with compactness and kernels.

Definition 55 *The function $Q : S \times S \to \Re$ is* **bilinear** *if $\forall\ x, y, z \in S$ and $\forall\ c \in \Re$:*
i) $Q(x + y, z) = Q(x, z) + Q(y, z)$,
ii) $Q(x, y + z) = Q(x, y) + Q(x, z)$,
iii) $Q(cx, y) = cQ(x, y)$, and
iv) $Q(x, cy) = cQ(x, y)$.

Definition 56 *The normed linear space $((S, +, \cdot), \|\ \cdot \|)$ is an* **inner product space** *if there is a bilinear function $Q : S \times S \to \Re$ so that*
i) $Q(x, y) = Q(y, x)\ \forall\ x, y \in S$ and
ii) $Q(x, x) = \|x\|^2\ \forall\ x \in S$.

Notation: Typically an innerproduct, $Q(x, y)$, is written $\langle x, y \rangle$.

Theorem 57 *If $((S, +, \cdot), \|\ \cdot \|)$ is an inner product space then the function Q above is unique.*

Definition 58 *Suppose $H = ((S, +, \cdot), \|\ \cdot \|)$ is an inner product space and $T \in L(H, H)$. The statement that T is* **compact** *means that if $x_1, x_2, \ldots$ is a bounded sequence in H, then $Tx_1, Tx_2, \ldots$ has a convergent subsequence.*

Definition 59 *If H is an inner product space and $T \in L(H,H)$ then T is* **symmetric** *means $\langle Tx, y\rangle = \langle x, Ty\rangle \ \forall\ x, y \in H$.*

Definition 60 *If H is an inner product space and $T \in L(H,H)$ then T is* **non-negative** *means $\langle Tx, x\rangle \ \geq 0 \ \forall\ x \in H$.*

Theorem 61 *If H is an inner product space, T is a symmetric member of $L(H,H)$, α and β are distinct eigenvalues of T, and x, y are the eigenvectors associated with α and β respectively, then $\langle x, y\rangle = 0$.*

Theorem 62 *Suppose H is an inner product space, T is a symmetric member of $L(H,H)$, and $Tx = \lambda x$ for some number λ and some $x \in H,\ x \neq 0$. If $y \in H$ and $\langle x, y\rangle = 0$ then $\langle x, Ty\rangle = 0$.*

Theorem 63 *(Cauchy, Schwartz, Bunakowski inequality) If H is an inner product space then*

$$\langle x, y\rangle^2 \leq \|x\|^2\|y\|^2 \ \forall\ x, y \in H.$$

Hint: Find t so that $\langle y - tx, x\rangle = 0$ and rewrite.

Note: From this point forward, we will suppose H is an inner product space and $T \in L(H,H)$ is symmetric, non-negative.

Definition 64 $\mathbf{N_T} = \inf\{\langle Tx, x\rangle : \|x\| = 1\}$.

Definition 65 *The* **operator norm** *on T is denoted by $|T|$ and is the smallest number M satisfying $\|Tx\| \leq |M|\,\|x\|$ all $x \in H$. is Equivalently, $|T| =$* $\sup\{\|Tx\| : \|x\| = 1\}$.

Theorem 66 $N_T \leq |T|$.

Theorem 67 *If $\lambda \in \Re,\ \lambda \neq 0,\ x \in H$, then*

$$\|Tx\|^2 = \frac{1}{4}\left[\langle T(\lambda x + 1/\lambda Tx), (\lambda x + 1/\lambda Tx)\rangle - \langle T(\lambda x - 1/\lambda Tx), (\lambda x - 1/\lambda Tx)\rangle\right].$$

Hint: Use bilinearity of inner product.

Theorem 68 $\|Tx\|^2 \leq \frac{N_T}{4}[\|\lambda x + 1/\lambda Tx\|^2 + \|\lambda x - 1/\lambda Tx\|^2],\ \forall\ x \in H,\ \lambda \in \Re,\ \ \lambda \neq 0.$

Hint: Simplify to: $\frac{N_T}{2}[\lambda^2\|x\|^2 + 1/\lambda^2\|Tx\|^2]$ and pick λ to minimize.

Theorem 69 *Show $|T| \leq N_T$ and conclude $|T| = N_T$.*

Theorem 70 *If T is a compact, symmetric, non-negative member of $L(H,H)$ for some inner product space H, then $|T|$ is an eigenvalue of T.*

Theorem 71 *If T is a compact, symmetric member of $L(H,H)$, then T has an eigenvalue λ so that $|\lambda| = |T|$.*

Definition 72 *If $X = (S, +, \cdot)$ is a (real or complex) vector space and $M \subset S$, the statement that M is* **convex** *means that if $x, y \in M$ then $tx + (1-t)y \in M$ $\forall\ t \in [0,1]$.*

Definition 73 *If $X = ((S, +, \cdot), \|\cdot\|)$ then $M \subset S$ is* **complete** *means that every Cauchy sequence in M converges to a point in M.*

Definition 74 *A* **Hilbert space** *is a complete inner product space.*

Theorem 75 *If H is an inner product space, M is a complete convex subset of $H, x \in H$, and $x \notin M$, then there is a unique point y of M such that*

$$\|x-y\| < \|x-w\| \ \forall\ w \in M,\ w \neq y.$$

Hint: Show there can't be *two* points y_1 and y_2 so that $\|x - y_1\| \leq \|x - w\|$ for all $w \in M$ *and* $\|x - y_2\| \leq \|x - w\|$ for all $w \in M$.

Definition 76 *Suppose H is an inner product space. The statement that P is an* **orthogonal projection** *on H means that there is a subspace S of H so that if $x \in H$, then*

$$Px \in S \text{ and } \|x - Px\| < \|x - y\| \ \forall\ y \in S,\ y \neq Px.$$

Theorem 77 *Suppose H is a Hilbert space. If P is an orthogonal projection on H, then*
i) $P \in L(H,H)$,
ii) $\langle Px, y\rangle = \langle x, Py\rangle \ \forall\ x, y \in H$ and
iii) $P^2 = P$.

Theorem 78 *Prove the converse to the previous theorem.*

Theorem 79 *Suppose that H is an inner product space and T is a symmetric, compact member of $L(H,H)$. If $\lambda \neq 0$, then the set $\{x \in H : Tx = \lambda x\}$ is finite dimensional. Equivalently, the eigenspace of T for the eigenvalue λ is finite dimensional.*

Theorem 80 *Prove Bessel's identity. Suppose H is an inner product space and $\phi_1, \phi_2, \ldots, \phi_n$ is an orthonormal sequence of members of H. If $c_1, c_2, \ldots, c_n \in \Re$ and $x \in H$, then*

$$\|x - (c_1\phi_1 + \cdots + c_n\phi_n)\|^2 = \|x\|^2 - \sum_{i=1}^{n} \langle x, \phi_i \rangle^2 + \sum_{i=1}^{n} (c_i - \langle x, \phi_i \rangle)^2.$$

Hint: Theorem 70.

Theorem 81 *Under the hypothesis of Theorem 79, if $\lambda \rangle 0$ there exists at most finitely many eigenvalues of T with magnitude greater than λ.*

Note: T can have infinitely many eigenvalues.

Definition 82 *The* **dual space**, *H^*, of the Hilbert space, H, is the collection of all linear functionals on H.*

Theorem 83 *If H is a Hilbert space and $f \in H^*$, then there is a unique point $y \in H$ so that $f(x) = \langle x, y \rangle \ \forall \ x \in H$.*

Definition 84 *If H is a Hilbert space and $T \in L(H, H)$ then the* **null space** *of T is defined by $N(T) = \{x : Tx = 0\}$.*

Note: $N(T)$ is a subspace of H.

Theorem 85 *Suppose each of H and K is a Hilbert space and $T \in L(H, K)$. Then there is a unique member T^* of $L(K, H)$ so that $\langle Tx, y \rangle_K = \langle x, T^*y \rangle_H \ \forall \ x \in H, \ y \in K$.*

Theorem 86 *Suppose each of H and K is a Hilbert space and $T \in L(H, K)$. Then $N(T) = \Re(T^*)^\perp$ and $N(T)^\perp = (R(T^*)^\perp)^\perp = \overline{R(T^*)}$.*

Problem 87 *(Suburban Problem) Divide a square grid into n by n pieces and attach a number to each grid point along boundary. Given the values of grid points on the boundary, pick initial values for all interior grid points, then assign values to each interior grid point by averaging its nearest neighbors. Iterate. Write a code for this elliptic problem.*

Problem 88 *Prove iteration in suburban problem goes to a unique solution.*

Problem 89 *Let u be a real valued continuously differentiable function on the square disk with corners at (0,0) and (1,1) such that $u_1 = u_2$ and $u(0, y) = f(y), 0 \leq y \leq 1$. Suppose $x \in (0, 1]$. Find an expression for the value of u at $(x, 0)$.*

Problem 90 *Divide [0,1] into n pieces of equal length and [0,x] into n pieces of equal length. Replace the partial differential equation in Problem 89 with a difference equation. Solve the difference equation to arrive at a value at* $(x,0)$*. (Discover the Bernstein polynomials).*

Definition 91 *Let* $p : [a,b] \to \Re$ *be a continuous function.*
i) $L_m y = y'' - py$ *if* $y \in C^{(2)}([a,b])$ *and* $y(a) = y'(a) = y(b) = y'(b) = 0$*. (minimal operator)*
ii) $L_M y = y'' - py$ *if* $y \in C^{(2)}([a,b])$ *(maximal operator)*
iii) $Ly = y'' - py$ *if* $y \in C^{(2)}([a,b])$ *and*
$A_1 \begin{pmatrix} y(a) \\ y'(a) \end{pmatrix} + A_2 \begin{pmatrix} y(b) \\ y'(b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,
where not both A_1 *and* A_2 *are zero matrices.*

Definition 92 *Let* $L^2_{[a,b]}$ *denote the set of all square [Lebesgue] integrable functions with inner product,* $\langle f, g\rangle = \int_a^b fg$*. Two such functions are considered equivalent if they differ only on a set of measure zero.*

Definition 93 *If* H *is a Hilbert space and* $T \in L(H,H)$*, then the* **adjoint** *of* T *is the operator,* T^**, with domain all* $y \in H$ *for which there is* $z \in H$ *such that* $\langle Tx, y\rangle = \langle x, z\rangle$ *for all* x *in the domain of* T*. For each such* y*,* $T^* y = z$*.*

Theorem 94 L_M *and* L_m *are adjoints.*

Theorem 95 *Suppose that for each* $g \in C_{[a,b]}$ $\exists$ *unique* $y \in Dom(L)$ *such that* $Ly = g$*. Then* $\langle Lf, g\rangle = \langle f, Lg\rangle \ \ \forall\ f, g\ \in Dom(L) \iff det(A_1) = det(A_2)$*.*

Problem 96 *Determine* $Range(L_m)$*.*

Problem 97 *If* $g \in C_{[a,b]}$ *find* $y \in Dom(L_M)$ *such that* $L_M y = g$ *and* $\|y\|$ *is minimum.*

Hint: Pick two numbers, (c,d), so that if y satisfies $y(a) = c, y'(a) = d$ and $y'' - py = q$, then $\|y\|$ is minimum as compared to $\|z\|$ for any other solution, z, to $z'' - pz = g$.

Problem 98 *Suppose for each* $g \in C_{[a,b]}$*,* T_g *denotes the element* y *as in Problem 97. Investigate* T^* *for* T *as in Problem 97.*

Theorem 99 $H = L^2_{[0,1]} \times L^2_{[0,1]}$ *is a Hilbert space under the inner product,*$\langle f, g\rangle = \sqrt{\int_0^1 f^2 + (f')^2}$*. and* $\{\begin{pmatrix} f \\ f' \end{pmatrix} : f \in C^1_{[0,1]}\}$ *is a subspace of* H*.*

Theorem 100 *The closure in* $L_2 \times L_2$ *of* $\{\begin{pmatrix} f \\ f' \end{pmatrix} : f \in C^1_{[0,1]}\}$ *is a function. I.e. No two pairs have the same first term and distinct second terms.*

Lemma 101 *If* $\begin{pmatrix} f_1 \\ f_1' \end{pmatrix}, \begin{pmatrix} f_2 \\ f_2' \end{pmatrix}, \cdots$ *converges in* $L^2_{[0,1]} \times L^2_{[0,1]}$ *to* $\begin{pmatrix} f \\ g \end{pmatrix}$ *then* $f_1, f_2, \ldots$ *converges uniformly (everywhere) to* f*. Thus* f *is continuous.*

Problem 102 *If each of* H *and* K *is a Banach space,* $\begin{pmatrix} r \\ s \end{pmatrix} \in H \times K$*, and* $T \in L(H,K)$*, find the unique point* $x \in H$ *so that* $\phi(x) = \frac{1}{2}\| \begin{pmatrix} x \\ Tx \end{pmatrix} - \begin{pmatrix} r \\ s \end{pmatrix} \|^2$ *is minimum.*

Problem 103 *Write* $(py')' - qy = g$ *as a system where* p *is continuous and positive on [a,b], letting* $v = py'$*. Generalize many of the above theorems to this setting.*

Problem 104 *Method of Lines on the heat equation. Solve system numerically:* $Y_i'(t) = Y_{i+1}(t) - 2Y_i(t) + Y_{i-1}(t)]/h^2 \ \forall\ i = 1, 2, \ldots, n-1, h = 1/n$*, where* $Y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$.

Theorem 105 *(Jordan Normal Form theorem) If* A *is a linear transformation from a complex finite dimensional space onto itself there always exists a basis of eigenvectors and generalized eigenvectors.*

Problem 106 *Suppose* $Az = \lambda z$ *and* z *is a generalized eigenvector (for eigenvalue* λ*) of index 2.* $e^{tA}z = \ldots.$

Definition 107 *Suppose that each of* X *and* Y *is a Banach space,* F *is a function, the domain of* F *is a subset of* X*, and the range of* F *is a subset of* Y*. The statement that* F *is* **Fréchet differentiable** *at the point* $x \in X$ *means that:*
i) $\{q \in X : \|q - x\| < r\} \subset Dom(F)$ *for some* $r\rangle 0$ *and*
ii) there exists $T \in L(X,Y)$ *such that if* $\epsilon\rangle 0$ *then* $\exists\ \delta\rangle 0$ *satisfying:*

$$\frac{\|F(y) - [F(x) + T(y-x)]\|_Y}{\|y - x\|} < \epsilon$$

provided $y \in X$ *and* $0 < \|y - x\| < \delta$*. We denote* T *by* $F'(x) = T$ *and* T *is unique.*

Definition 108 *Suppose that each of* X *and* Y *is a Banach space,* F *is a function, the domain of* F *is a subset of* X*, and the range of* F *is a subset of* Y*.*

The statement that F is **twice differentiable** *at $x \in X$ means that if $h \in X$ and $g_h(y) = F'(y)h \ \forall \ y \in Dom(F')$ then g_h is differentiable at x. If F is twice differentiable at x then $F''(x)(h,k) = (g_h'(x))k \ \forall \ h,k \in X$.*

Note: F'' is bilinear but not necessarily linear.

Theorem 109 *If the domain of F'' contains an open subset Ω f X and F'' is continuous in an appropriate sense, then $F''(x)$ is a* **symmetric** *bilinear function for all $x \in \Omega$.*

Problem 110 *Define higher Frećhet derivatives $F^{(n)}$ for $n > 2$. Prove a symmetry result under the continuity hypothesis above. Define a Taylor's formula,*

$F(x) = F(a) + F'(a)(x-a) + \frac{1}{2!}F''(a)(x-a)^2 + \ldots + + \frac{1}{n-1}\int_0^1 F^{(n-1)}(a + s(x - a))(x-a)^{n-1}ds + \frac{1}{n}\int_0^1 F^{(n)}(a+s(x-a))(x-a)^n ds.$

If $F^{(n)}$ exists and is continuous for all y so that $\|y-a\| < r$ then for all x so that $\|x-a\| < r$,

$F(x) = F(a) + F'(a)(x-a) + \frac{1}{2!}F''(a)(x-a)^2 + \ldots + + \frac{1}{n-1}\int_0^1 F^{(n-1)}(a + s(x - a))(x-a)^{n-1}ds + \frac{1}{n}\int_0^1 F^{(n)}(a+s(x-a))(x-a)^n ds.$

Chapter 6

Foundations, Ochoa

6.1 Introduction

This is a self-contained course in mathematical foundations. It requires no formal prequesite. At Hardin-Simmons University, we require two semesters of calculus before a student may take this course; however, this is just to ensure that we have "serious" students enroll.

I have taken great pains to ensure that this course actually covers some meaningful mathematics, and is not just a theorem proving course. The topics include logic, set theory, number theory, and functions and relations. The topics are integrated by design. Quite frankly, I am not very happy with the way I have handled the logic rules and set theory. In future revisions, I intend to integrate the logic rules and set theory even more.

Anyone who decides to use these notes is encouraged to adapt them to his specific needs. I recommend that these notes be used only as a basis for a developing a course.

6.2 Theorem Sequence

Math 4350
Seminar in Mathematics
Mathematics Structures
Spring, 1999

Instructor: Dr. James Ochoa
Office: SR-B5
Telephone: 670-1388

Office Hours: Monday through Thursday, 1:00 - 3:30, or by appointment. You are welcome to stop by and visit me any time my office door is open.
No textbook is required. You will need a loose-leaf notebook and a hole-punch. You are responsible for the notes I give you.

Grading Policy.

I will assign problems. Students will work the problems and present well-written solutions in class. In addition, I will also assign problems to be turned in. Students will be evaluated on both the quality and quantity of correct solutions. The course grade is subjective.

Most of the problems will involve proving statements.

Attendance Policy.

Attendance is required. Five points will be deducted from the final grade for each unexcused absence. The first three absences are automatically excused. In order for an absence to be excused, proper documentation must be provided. In the case of illness, a note from a doctor or school medical personnel is required. In the case of a funeral, a death notice is required. In order for school sponsored activities to be excused, the student must meet with me before the absence. Any student who is absent eleven or more times, excused or unexcused, will fail the course.

You are to present only your work. The only exception is if I help you. Presenting anyone else's work is considered cheating. If you understand this, sign beside this paragraph.

The above is the course syllabus.

We begin with some formal rules of logic. A statement is a sentence, which in a given context, is either true or false.

Logic Rule 1 *No unstated assumption may be used in a proof.*

Logic Rule 2 *Let p and q be statements. If the compound statement "p and not q" implies a contradiction, then the statement "If p, then q" is true.*

We may use Logic Rule 2 to prove the statement in the following example.

Example 3 *Let n be a positive integer. If 9 divides n, then 3 divides n.*

Proof. Suppose that 9 divides n and 3 does not divide n. Since 3 does not divide n, neither does 3^2. Since $3^2 = 9$, it follows that 9 cannot divide n. This is a contradiction, since 9 divides n. Therefore, if 9 divides n, then 3 divides n.

Logic Rule 4 *Let p be a statement. The negation of the statement "Not p" means the same as p.*

Logic Rule 5 *Let p and q be statements. The negation of the statement "If p, the q" means the same as "p and not q.*

Logic Rule 6 *Let p and q be statements. The negation of the statement "p and q" means the same as "Not p or not q." The negation of the statement "p or q" means the same as "Not p and not q."*

Let $p(x)$ be a sentence about x (in this case x is called a variable). We say that z is in the domain of p if the sentence $p(z)$ is a statement. For example, let $p(x)$ be the sentence "$x + 3 = 7$." Then 6 is in the domain of p since $p(6)$ is the (false) statement "$6 + 3 = 7$." On the under hand w is not in the domain of p, since we cannot, in the current context, determine whether "$w + 3 = 7$" is true or false.

Let $p(x)$ be a sentence about x. The sentence "For all x, $p(x)$" is a statement. The expression "for all" is called the universal quantifier. It is understood that the phrase "for all x" refers only to x in the domain of $p(x)$. The sentence "There exists an x such that $p(x)$" is a statement. The expression "there exists" is called the existential quantifier. It is understood that the phrase "there exists an x" refers to some x in the domain of $p(x)$.

Logic Rule 7 *Let $p(x)$ be a sentence about x. The negation of the statement "For all x, $p(x)$" means the same as "There exists an x such that not $p(x)$."*

Logic Rule 8 *Let $p(x)$ be a sentence about x. The negation of the statement "There exists an x such that $p(x)$" means the same as "For all x, not $p(x)$."*

Logic Rule 9 *Let p and q be statements. Suppose the statements "If p, then q" and p are steps in a proof. Then q is a valid step.*

Logic Rule 10 *Let p, q, and r be statements. The following are all true statements:*

1. *If p, then p or q.*
   If q, then p or q.
2. *If p and q, then p.*
   If p and q, then q.
3. *If "Not q" implies "Not p," then p implies q.*
   If p implies q, then "Not q" implies "Not p".
4. *If p implies q and q implies r, then p implies r.*

Logic Rule 11 *For every statement p, either p or "Not p" is true.*

Logic Rule 12 *Let p, q, and r be statements. Let 1 denote a true statement and let 0 denote a false statement. Then:*

1. *The statement "p and q" means the same as "q and p,"*
2. *The statement "p or q" means the same as "q or p"*
3. *The statement "p and q; and r" means the same as "p; and q and r."*
4. *The statement "p or q; or r" means the same as "p; or q or r"*
5. *The statement "p and q; or r" means the same as "p or r; and q or r,"*
6. *The statement "p or q; and r" means the same as "p and r; or q and r,"*
7. *The statement "p or 0" means the same as p,*
8. *The statement "p and 0" is false,*
9. *The statement "p or 1" is true,*
10. *The statement "p and 1" means the same as p,*
11. *The statement "p or not p" is true, and*
12. *The statement "p and not p" is false.*

The following will remain undefined: *set, is a member of, collection, a sentence about,statement, natural number*, and *integer.*

Let A be a set. If x is a member of A, we write $x \in A$. If x is not a member of A, we write $x \notin A$.

Definition 13 *Let A and B be sets. We say that A is a subset of B, and write $A \subseteq B$, if the following statement is true:*

If $x \in A$, then $x \in B$.

For example, let $A = \{1, 3, 5\}$ and let $B = \{1, 2, 3, 4, 5\}$. Then $A \subseteq B$.

Let S be a set and let $p(x)$ be a sentence about members x of S. We write $\{x|p(x)\}$ to denote the collection of all $x \in S$ such that $p(x)$ is a true statement. In this case, $p(x)$ is called a condition.

Axiom 14 *Let S be a set and let $p(x)$ be a sentence about members x of S. Then $\{x|p(x)\}$ is a subset of S.*

Definition 15 *Let A and B be sets. We say that A and B are equal, and write $A = B$ if $A \subseteq B$ and $B \subseteq A$.*

For example, let $A = \{2, 4, 6, 8\}$ and let $B = \{8, 6, 4, 2\}$. Then A and B are equal.

We write $\mathbf{N}$ to denote the set of natural numbers. We write $\mathbf{Z}$ to denote the set of integers.

Axiom 16 *The set of natural numbers is a subset of the set of integers.*

The empty set, $\emptyset$, is the set with no members.

Proposition 17 *Let S be a set. Then $\emptyset \subseteq S$ and $S \subseteq S$.*

Definition 18 *Let S be a set. The power set of S, $\mathcal{P}(S)$, is the collection of all subsets of S. In this context, the set S is called the universal set.*

Axiom 19 *Let S be a set. Then $\mathcal{P}(S)$ is a set.*

Problem 20 *Let $S = \{2, 3, 4\}$. List the members of $\mathcal{P}(S)$.*

Definition 21 *Let S be a set. Let A and B be subsets of S.*

1. *The intersection of A and B is the set*

$$A \bigcap B = \{x | x \in A \text{ and } x \in B\}.$$

2. *The union of A and B is the set*

$$A \bigcup B = \{x | x \in A \text{ or } x \in B\}.$$

3. *The complement of* A *is the set*

$$\sim A = \{x | x \in S \text{ and } x \notin A\}$$

Proposition 22 *Let* S *be a set. Let* A *and* B *be subsets of* S*. Then* $A \bigcap B$, $A \bigcup B$, *and* $\sim A$ *are all subsets of* S.

Proposition 23 *Let* A *and* B *be sets. Then* $A \bigcap B$ *is a subset of* A.

Proof. Let x be a member of $A \bigcap B$. Then $x \in A$ and $x \in B$. In particular, $x \in A$. It follows that if $x \in A \bigcap B$, then $x \in A$. Therefore, $A \bigcap B \subseteq A$. □

Proposition 24 *Let* A *and* B *be sets. Then* A *is a subset of* $A \bigcup B$.

Proposition 25 *Let* A, B, *and* C *be sets. If* $A \subseteq B$ *and* $B \subseteq C$, *then* $A \subseteq C$.

Proposition 26 (Commutative Property) *Let* S *be a set and let* A *and* B *be subsets of* S*. Then*

1. $A \bigcup B = B \bigcup A$ *and*
2. $A \bigcap B = B \bigcap A$.

Proposition 27 (Associative Property) *Let* S *be a set and let* A, B, *and* C *be subsets of* S*. Then*

1. $(A \bigcap B) \bigcap C = A \bigcap (B \bigcap C)$ *and*
2. $(A \bigcup B) \bigcup C = A \bigcup (B \bigcup C)$.

Proposition 28 (Distributive Property) *Let* S *be a set. Let* A, B, *and* C *be subsets of* S*. Then*

1. $(A \bigcap B) \bigcup C = (A \bigcup C) \bigcap (B \bigcup C)$ *and*
2. $(A \bigcup B) \bigcap C = (A \bigcap C) \bigcup (B \bigcap C)$.

Proposition 29 *Let* S *be a set and let* A *be a subset of* S*. Then*

1. $A \bigcup \emptyset = A$ *and*
2. $A \bigcap \emptyset = \emptyset$.

Proposition 30 *Let* S *be a set and let* A *be a subset of* S*. Then*

1. *$A \bigcup S = S$ and*
2. *$A \bigcap S = A$.*

Proposition 31 *Let S be a set and let A be a subset of S. Then*

1. *$A \bigcup \sim A = S$ and*
2. *$A \bigcap \sim A = \emptyset$.*

Proposition 32 (DeMorgan's Laws) *Let S be a set and let A and B be subsets of S. Then*

1. *$\sim(A \bigcap B) = \sim A \bigcup \sim B$, and*
2. *$\sim(A \bigcup B) = \sim A \bigcap \sim B$.*

Proposition 33 *Let A, B, and C be sets such that $A = B$ and $B = C$. Then $A = C$.*

Logic Rule 34 *Let $p(x)$ be a sentence about x and let q be a statement. Then*

1. *The statement "(there exists an x such that $p(x)$) and q" means the same as "there exists an x such that ($p(x)$ and q)."*
2. *The statement "(for all x, $p(x)$) or q" means the same as "for all x, ($p(x)$ or q)" and*

Let Δ and S be sets. For each δ in Δ, suppose A_δ is a subset of S. We define the following sets:

$$\bigcup_{\delta \in \Delta} A_\delta = \{x | x \in S \textit{ and there exists a } \delta \in \Delta \textit{ such that } x \in A_\delta, \}$$

and

$$\bigcap_{\delta \in \Delta} A_\delta = \{x | x \in S \textit{ and, for all } \delta \in \Delta,\ x \in A_\delta\}.$$

Note that $\bigcup_{\delta \in \Delta} A_\delta$ and $\bigcap_{\delta \in \Delta} A_\delta$ are both subsets of S.

Proposition 35 (Generalized Distributive Property for Sets: Homework) *Let Δ and S be sets. Let B be a subset of S. For each δ in Δ, suppose A_δ is a subset of S. Then*

$$(\bigcup_{\delta \in \Delta} A_\delta) \bigcap B = \bigcup_{\delta \in \Delta} (A_\delta \bigcap B),$$

and

$$(\bigcap_{\delta \in \Delta} A_\delta) \bigcup B = \bigcap_{\delta \in \Delta} (A_\delta \bigcup B).$$

Proposition 36 (Generalized DeMorgan's Laws for Sets: Homework) *Let Δ and S be sets. For each δ in Δ, suppose A_δ is a subset of S. Then*

$$\sim\left(\bigcap_{\delta\in\Delta} A_\delta\right) = \bigcup_{\delta\in\Delta}(\sim A_\delta),$$

and

$$\sim\left(\bigcup_{\delta\in\Delta} A_\delta\right) = \bigcap_{\delta\in\Delta}(\sim A_\delta).$$

Axiom 37 (Equality Properties for Integers) *Let a, b, and c be integers. Then*

Reflexive Property $a = a$,

Symmetric Property *If $a = b$, then $b = a$, and*

Transitive Property *If $a = b$ and $b = c$, then $a = c$.*

Axiom 38 (Closure Property for Integers) *If a and b are integers, then so are $a + b$ and ab.*

Axiom 39 (Commutative Property for Integers) *If a and b are integers, then $a + b = b + a$ and $ab = ba$.*

Axiom 40 (Associative Property for Integers) *If a and b are integers, then $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.*

Axiom 41 (Distributive Property for Integers) *If a, b, and c are integers, then $(a + b)c = ac + bc$.*

Axiom 42 (Identity Elements for Integers) *If a is an integer, then $a+0 = a$ and $a \times 1 = a$.*

Axiom 43 (Additive Inverse of Integers) *For each integer a, there exists an integer b such that $a + b = 0$.*

The integer b in Axiom 43 is called an additive inverse of a.

Axiom 44 (Equation Rules for Integers) *Let a, b, and c be integers such that $a = b$. Then $a + c = b + c$ and $ac = bc$.*

Axiom 45 (Cancellation Law for Integers) *If a, b, and c are integers such that $ac = bc$ and $c \neq 0$, then $a = b$.*

Proposition 46 *Let a, b, and c be integers. Then $a(b+c) = ab + ac$.*

Proposition 47 *If a is an integer, then $0a = 0$.*

Proof. Let a be an integer. By the Identity Axiom, $0 + 0 = 0$. Thus, by the Equation Rules, $(0 + 0)a = 0a$. Using the Distributive Property, $0a + 0a = 0a$. Let b be an integer such that $0a + b = 0$. Then $(0a + 0a) + b = 0a + b$. Using the Associative Property, $0a + (0a + b) = 0a + b$. Thus $0a + 0 = 0$; that is, $0a = 0$. □

Proposition 48 *Let a. b, and c be integers. If $a + b = 0$ and $a + c = 0$, then $b = c$.*

Proposition 48 tells us that each integer has a unique additive inverse. Let a an be integer. We will write $-a$ to denote the additive inverse of a. Thus $a + (-a) = 0$.

Definition 49 *Let a and b be integers. We define $a - b$ by*

$$a - b = a + (-b).$$

Definition 50 *Let a and b be integers. We write $a < b$ if $b - a$ is a natural number. If $a < b$, we also write $b > a$. We also write $a \leq b$, or $b \geq a$, if either $a < b$ or $a = b$.*

Proposition 51 *Let a be an integer. Then a is a natural number if, and only if, $a > 0$*

Axiom 52 (Closure Property for Natural Numbers) *If a and b are natural numbers, then so are $a + b$ and ab.*

Axiom 53 (Trichotomy Law for Integers) *For every integer a, either $a > 0$, $a = 0$, or $a < 0$.*

Axiom 54 *The integer 1 is a natural number. Moreover, if n is any natural number, then $1 \leq n$.*

Proposition 55 *The integer 0 is its own additive inverse.*

Proposition 56 *Let a be an integer. If $-a = a$, then $a = 0$.*

Proposition 57 *Let a be an integer. Then $-1a = -a$.*

Proposition 58 *Let a be an integer. Then $-(-a) = a$*

Proposition 59 *Let a and b be integers. Then $(-a)(-b) = ab$.*

Proposition 60 *Let a and b be integers. If $ab = 0$, then $a = 0$ or $b = 0$.*

Proof. Assume that $ab = 0$. Either $a = 0$ or $a \neq 0$. If $a = 0$, we are done! Suppose that $a \neq 0$. Then $ab = 0$ and $0 = 0a$. Thus, $ab = 0a$; that is, $ba = 0a$. By the Cancellation Law, we have $b = 0$. □

Proposition 61 *Let a, b, and c be integers. Suppose $a < b$ and $c > 0$. Then, $ac < bc$.*

Proposition 62 *Let a, b, and c be integers. Suppose $a < b$ and $c < 0$. Then, $ac > bc$.*

Hint. Show that $-c > 0$.

A set of integers A is said to have a least element l if l is a member of A and for all x in A, $l \leq x$.

Axiom 63 (Well-ordering Property for Natural Numbers) *Every nonempty subset of the natural numbers has a least element.*

Theorem 64 (First Principle of Mathematical Induction) *Let S be a subset $\mathbf{N}$ which satisfies the following two conditions:*

1. *$1 \in S$, and*
2. *If $n \in S$, then $n + 1 \in S$.*

Then $S = \mathbf{N}$.

Hint. Let T be the set of natural numbers not in S. Either $T = \emptyset$ or $T \neq \emptyset$. If $T = \emptyset$, we are done!

Theorem 65 (Second Principle of Mathematical Induction) *Let S be a subset of $\mathbf{N}$ which satisfies the following two conditions:*

1. *$1 \in S$, and*
2. *For each natural number n, if the set $\{k | k \in \mathbf{N}$ and $k < n\} \subseteq S$, then $n \in S$.*

Then, $S = \mathbf{N}$.

Theorem 66 *For each natural number n, let $P(n)$ be a statement. Suppose the following two statements are true:*

1. *$P(1)$ is a true statement, and*
2. *For each natural number n, $P(n)$ implies $P(n+1)$.*

Then $P(n)$ is true for all n.

Theorem 67 *For each natural number n, let $P(n)$ be a statement. Suppose the following two statements are true:*

1. *$P(1)$ is a true statement, and*
2. *For each natural number n, if $P(k)$ is true for every natural number $k < n$ then $P(n)$ is also true.*

Then $P(n)$ is true for all n.

Problem 68 *Prove that for each natural number n,*

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Problem 69 *Prove that for each natural number n,*

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Definition 70 *Let a, b, and c be integers. We say that a divides b, and write $a \mid b$, if there is an integer c such that $b = ac$. If a does not divide b, we write $a \not| b$.*

Proposition 71 *Let a, b, and c be integers. If $a \mid b$ and $b \mid c$, then $a \mid c$.*

Proposition 72 *Let a, b, c, m, and n be integers. If $c \mid a$ and $c \mid b$, then $c \mid (am + bn)$.*

Lemma 73 *Let a and b be integers such that $0 \leq a < b$. If $b \mid a$, then $a = 0$.*

Theorem 74 (Division Algorithm) *Let a and b be integers such that $b > 0$. Then there exist unique integers q and r such that $0 \leq r < b$ and $a = bq + r$.*

Hint. You must show (1) that q and r exist and (2) that q and r are unique. To prove part (1), let S be the set of all nonnegative integers of the form $a - bk$, where k is an integer. Show that $a \in S$ or $a - ab \in S$. Thus S is not empty. Let r be the least element of S. Continue .... To prove part (2), suppose that $a = bq + r$, $a = b' + r'$, and $0 \leq r' \leq r < b$. Do some algebra.

Let n be and integer. If $2 \mid n$, then we say n is even. If $2 \not| n$, then we say n is odd.

Proposition 75 *Let a and b be integers with $b > 0$. Let $a = bq + r$ such that $0 \leq r < b$. Then $b \mid a$ if, and only if, $r = 0$.*

Theorem 76 (Recursion Principle) *Let S be a set. For each natural number n let d_n be a member of S. Suppose the following two conditions are satisfied:*

1. *d_1 is defined, and*
2. *For each natural number n, if d_n is defined then d_{n+1} is also defined.*

Then d_n is defined for each natural number n.

Definition 77 *Let n be a natural number. We define $0!$ and $n!$ by*

$$0! = 1,$$

and

$$n! = n(n-1)!.$$

Definition 78 *Let p be a natural number such that $p > 1$. We say that p is prime if the only natural numbers which divide p are 1 and p. If n is a natural number, $n > 1$, and n is not prime, then we say that p is composite.*

Proposition 79 *Every natural number greater that 1 has a prime divisor.*

Hint. Let S be the set of natural numbers greater that 1 which do not have a prime divisor. We want S to be empty. Suppose S is not empty ....

Theorem 80 *For each natural number n, there is a prime p such that $p \geq n$.*

Theorem 80 asserts that there are infinitely many primes.

Proposition 81 (Homework) *Let n be a natural number. There are consecutive integers $m, m+1, m+2, \ldots, m+n-1$, none of which are prime.*

Conjecture 82 *Let n be an even natural number greater than 2. Then there exist primes p and q such that $n = p + q$.*

Definition 83 *Let a and b be integers such that $a \neq 0$ or $b \neq 0$. The greatest common divisor of a and b, $\gcd(a,b)$, is defined to be that largest natural number d such that $d \mid a$ and $d \mid b$. That is, if $d = \gcd(a,b)$, $l \mid a$, and $l \mid b$, then $l \leq d$. In addition, $\gcd(0,0)$ is defined to be 0.*

Observe that $\gcd(a,b) = \gcd(b,a)$.

Problem 84 *Calculate* $\gcd(-48, 36)$.

Definition 85 *Two integers a and b are said to be relatively prime if $\gcd(a,b) = 1$.*

Proposition 86 *Let a, b, d, m, and n be integers such that $d = \gcd(a,b)$, $a = dm$, and $b = dn$. If $d \neq 0$, then $\gcd(m,n) = 1$.*

Proposition 87 *Let a, b, and c be integers. Then $\gcd(a + bc, b) = \gcd(a,b)$.*

Theorem 88 *Let a and b be integers such that $a \neq 0$. Let $d = \gcd(a,b)$. Then there exist integers m and n such that $d = ma + nb$.*

Hint. Note that $d \neq 0$. Let $S = \{xa + yb | x, y \in \mathbf{Z} \text{ and } ax + by > 0\}$. Show that $S \neq \emptyset$. Let l be the least member of S. Write $l = ma + nb$ where $m, n \in \mathbf{Z}$. Use the division algorithm to show that $l \mid a$. Similarly, $l \mid b$. This one may take some work.

Proposition 89 *Let a, b, d, and k be integers. Assume that $d = \gcd(a,b)$, $k \mid a$, and $k \mid b$. Then $k \mid d$.*

Theorem 90 (Euclid, c. 350 B.C.) *Let a, b, q, and r be integers such that $0 \leq r < b$ and $a = bq + r$. Then*

$$\gcd(a, b) = \gcd(a, r).$$

Problem 91 (Homework) *Find* $\gcd(102, 222)$ *and* $\gcd(20785, 44350)$.

Lemma 92 *Let a, b, and c be natural numbers. If* $\gcd(a, b) = 1$ *and $a \mid bc$, then $a \mid c$.*

Lemma 93 *Suppose p is prime and a_1, a_2, ..., a_n are all natural numbers. If $p \mid a_1 a_2 \cdots a_n$, then there is a integer i such that $1 \leq i \leq n$ and $p \mid a_i$.*

Hint. You will need to use Theorem 67 applied to n, the number of factors a_1, a_2, ..., a_n, and Lemma 93. Show things work for $n = 1$ and $n = 2$. Assume that for $2 < k < n$ if $p \mid a_1 a_2 \cdots a_k$, then there is a integer i such that $1 \leq i \leq k$ and $p \mid a_i$. Prove that if $p \mid a_1 a_2 \cdots a_n$, then there is a integer i such that $1 \leq i \leq n$ and $p \mid a_i$.

We've got the biggie coming up next time.

Theorem 94 (The Fundamental Theorem of Arithmetic) *Let n be a natural number greater than 1. There exist a unique natural number k and unique primes $P_1, P_2, \ldots, P_k$ such that $P_1 \leq P_2 \leq \ldots \leq P_k$ and*

$$n = P_1 P_2 \ \cdots P_k.$$

We will prove this one in class.

The factorization $n = P_1 P_2 \ \cdots P_k$ in the Fundamental Theorem is called the prime factorization of n. Usually, we write the prime factorization of n in the form

$$n = \rho_1^{l_1} \rho_2^{l_2} \cdots \rho_m^{l_m}$$

where m is a natural number, $l_1, l_2, \ldots, l_m$ are natural numbers, $\rho_1, \rho_2, \ldots, \rho_m$ are primes, and $\rho_1 < \rho_2 < \cdots < \rho_m$.

Problem 95 *Write the prime factorization of* 15444.

Definition 96 *Let A and B be sets. We define the Cartesian Product of A and B, $A \times B$ to be the collection of all pairs (a, b) such that a is a member of A and b is a member of B. We call (a, b) an ordered pair.*

Axiom 97 *Let A and B be sets. Then $A \times B$ is a set.*

In light of Axiom 97 we may write

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}.$$

Axiom 98 *Let A and B be sets. Let (a, b) and (c, d) be members of $A \times B$. Then $(a, b) = (c, d)$ if, and only if, $a = c$ and $b = d$.*

Problem 99 *Let $A = \{a, b, c, d, e\}$ and $B = \{a, e, i, o, u\}$. List the members of $A \times B$.*

Definition 100 *Let A and B be sets. A subset f of $A \times B$ is said to be a function (or map) from A into B if the following two conditions are satisfied:*

1. *For each $a \in A$, there exists $b \in B$ such that $(a, b) \in f$, and*
2. *For each $a \in A$ and for all $b_1, b_2 \in B$, if $(a, b_1) \in f$ and $(a, b_2) \in f$, then $b_1 = b_2$.*

If f is a function from A into B, we write $f\colon A \longrightarrow B$. The set A is called the domain of f. If $(a, b) \in f$, we write $b = f(a)$. The set

$$\{b \in B | \text{ There exists } a \in A \text{ such that } b = f(a)\}$$

is called the range of f.

Problem 101 *Let* $A = \{a, b, c, d, e\}$ *and* $B = \{a, e, i, o, u\}$. *Let* $f = \{(a, i), (b, i), (c, a), (d, i), (e, e)\}$.

1. *Is* f *a function from* A *into* B*?*
2. *Find the domain of* f.
3. *Find the range of* f.
4. *Find* $f(d)$.

Problem 102 *Let* $A = \{a, b, c, d, e\}$ *and* $B = \{a, e, i, o, u\}$. *Let* $f = \{(a, e), (b, a), (c, o), (e, a)\}$. *Is* f *a function from* A *into* B*? Explain your answer.*

Problem 103 *Let* $A = \{a, b, c, d, e\}$ *and* $B = \{a, e, i, o, u\}$. *Let* $f = \{(a, e), (b, a), (c, o), (c, a), (d, u), (e, a)\}$. *Is* f *a function from* A *into* B*? Explain your answer.*

Problem 104 *Let* $A = \{a, b, c, d, e\}$ *and* $B = \{a, e, i, o, u\}$. *Let* $f = \{(a, a), (b, b), (c, c), (i, i), (d, d), (e, e)\}$. *Is* f *a function from* A *into* B*? Explain your answer.*

Definition 105 *Let* $f\colon A \longrightarrow B$ *be a function.* f *is defined to be one-to-one, 1–1, if the following condition is satisfied:*

> *For all* a_1 *and* a_2 *in* A *and* b *in* B, *if* $(a_1, b) \in f$ *and* $(a_2, b) \in f$, *then* $a_1 = a_2$.

Problem 106 *Let* $A = \{w, x, y, z\}$ *and* $B = \{a, e, i, o, u\}$. *Give examples of two functions from* A *into* B, *one of which is 1–1 and one that is not.*

Definition 107 *Let* $f\colon A \longrightarrow B$ *be a function.* f *is defined to be a function from* A *onto* B *if the following condition is satisfied:*

> *For each* b *in* B, *there is an* a *in* A *such that* $(a, b) \in f$; *that is,* $b = f(a)$.

Problem 108 *Let* $A = \{u, v, w, x, y, z\}$ *and* $B = \{a, e, i, o, u\}$. *Give examples of two functions from* A *into* B, *one of which is from* A *onto* B *and one that is not.*

Problem 109 *Let* $A = \{u, v, w, x, y, z\}$ *and* $B = \{a, e, i, o, u\}$. *Is there a 1–1 function from* A *into* B*? Explain your answer.*

Problem 110 *Let* $A = \{w, x, y, z\}$ *and* $B = \{a, e, i, o, u\}$. *Is there a function from* A *onto* B*? Explain your answer.*

Problem 111 *Let $A = \{a, b, c, d, e\}$ and $B = \{a, e, i, o, u\}$. Find a 1–1 function from A onto B.*

Definition 112 *Let $g\colon A \longrightarrow B$ and $f\colon B \longrightarrow C$ be functions. We define the composition of f and g, $f \circ g$, to be the set*

$\{(a, c) | a \in A,\ c \in C,$ *and there exists* $b \in B$ *such that* $(a, b) \in g$ *and* $(b, c) \in f\}$.

Proposition 113 *Let $g\colon A \longrightarrow B$ and $f\colon B \longrightarrow C$ be functions. Then $f \circ g$ is a function from A into C.*

Proposition 114 (Homework) *Let $g\colon A \longrightarrow B$ and $f\colon B \longrightarrow C$ both be 1–1 functions. Then $f \circ g$ is also a 1–1 function.*

Proposition 115 (Homework) *Let $g\colon A \longrightarrow B$ and $f\colon B \longrightarrow C$ be functions from A onto B and from B onto C respectively. Then $f \circ g$ is a function from A onto C.*

Definition 116 *Let $f\colon A \longrightarrow B$ be a function. Define f^{-1} by*

$$f^{-1} = \{(b, a) | (a, b) \in f\}.$$

The set f^{-1} is called the inverse of f.

Proposition 117 *Let $f\colon A \longrightarrow B$ be a 1–1 function from A onto B. Then f^{-1} is a 1–1 function from B onto A.*

Hint. You need to show that f^{-1} is a function, f^{-1} is 1–1, and f^{-1} is onto.

Proposition 118 (Homework) *Let $f\colon A \longrightarrow B$ be a 1–1 function from A onto B. Then $(f^{-1})^{-1} = f$.*

Proposition 119 *Let $f\colon A \longrightarrow B$ and $g\colon A \longrightarrow B$ be functions. Then $f = g$ if, and only if, for all a in A, $f(a) = g(a)$.*

Proposition 120 *Let $h\colon A \longrightarrow B$, $g\colon B \longrightarrow C$, and $f\colon C \longrightarrow D$ be functions. Then $f \circ (g \circ h) = (f \circ g) \circ h$.*

Definition 121 *Let A and B be sets, and let n be natural number.*

1. *We say that $|A| \leq |B|$ if there is a 1–1 function from A into B.*
2. *We say that $|A| = |B|$ if $|A| \leq |B|$ and $|B| \leq |A|$.*

3. *We say that* $|A| < |B|$ *if* $|A| \leq |B|$*, but there is not a 1–1 function from* B *into* A*.*
4. *We say that* $|A| = n$ *if there is a 1–1 function from* A *onto the set* $\{x | x \in \mathbf{N}$ *and* $x \leq n\}$*.*
5. *We define* $|\emptyset|$ *by* $|\emptyset| = 0$*.*

The notation $|A|$ is called the cardinality of A. Note that $|A| = |B|$ if, and only if, $|B| = |A|$. We say that A is a finite set if $|A| = n$. If A is not finite, we say that A is infinite.

We will not prove the next theorem.

Theorem 122 (The Schröder-Bernstein Theorem) *Let* A *and* B *be sets. Then* $|A| = |B|$ *if, and only if, there is a 1–1 function from* A *onto* B*.*

Proposition 123 *Let* A *be a set. Then* $|A| = |A|$*.*

Proposition 124 *Let* A*,* B*, and* C *be sets. If* $|A| \leq |B|$ *and* $|B| \leq |C|$*, then* $|A| \leq |C|$*. Moreover, if* $|A| = |B|$ *and* $|B| = |C|$*, then* $|A| = |C|$*.*

Definition 125 *Let* A *be an infinite set. We say that* A *is countable if* $|A| = |\mathbf{N}|$*. If* A *is not countable, we say that* A *is uncountable.*

Proposition 126 *The set* $\mathbf{Z}$ *is countable.*

Proposition 127 *The set* $\mathbf{Z} \times \mathbf{Z}$ *is countable.*

Proposition 128 *The set of even natural numbers,* $2\mathbf{N}$*, is countable.*

Proposition 129 *Let* A *be a countable set. Let* B *be an infinite subset of* A*. Then* B *is countable.*

Proof. Write $A = \{a_1, a_2, \ldots, a_n, \ldots\}$. Let i_1 be the least natural number such that $a_{i_1} \in B$. Let i_2 be the least natural number such that $a_{i_2} \in B \bigcap \sim\{a_{i_1}\}$. Let i_3 be the least natural number such that $a_{i_3} \in B \bigcap \sim\{a_{i_1}, a_{i_2}\}$. In general, let i_k be the least natural number such that $a_{i_k} \in B \bigcap \sim\{a_{i_1}, a_{i_2}, \ldots, a_{i_{k-1}}\}$. It follows that we may write $B = \{a_{i_1}, a_{i_2}, \ldots, a_{i_k} \ldots\}$. Define $f : \mathbf{N} \longrightarrow B$ by $f(k) = a_{i_k}$. Note that f is a 1–1 function from $\mathbf{N}$ onto B. Thus $|B| = |\mathbf{N}|$.

Axiom 130 (Principle of Addition) *Let* A *and* B *be sets such that* $A \bigcap B = \emptyset$*. Let* m *and* n *be natural numbers. If* $|A| = m$ *and* $|B| = n$*, then* $|A \bigcup B| = m + n$*.*

Axiom 131 (Principle of Multiplication) *Let* A *and* B *be sets. Let* m *and* n *be natural numbers. If* $|A| = m$ *and* $|B| = n$*, then* $|A \times B| = mn$*.*

Problem 132 (Homework) *Let* $A = \{a, b, c, d\}$ *and* $B = \{e, f, g, h, i\}$. *Find* $|A \bigcup B|$.

Problem 133 (Homework) *Let* $A = \{a, b, c, d\}$ *and* $B = \{b, d, e, f\}$. *Find* $|A \bigcup B|$.

Problem 134 (Homework) *Let* $A = \{a, b, c, d\}$ *and* $B = \{e, f, g, h, i\}$. *Find* $|A \times B|$.

Problem 135 (Homework) *Let* $A = \{a, b, c, d\}$ *and* $B = \{b, d, e, f\}$. *Find* $|A \times B|$.

Problem 136 (Homework) *Let* A *be a set. Find* $|A \times \emptyset|$.

Definition 137 *Let* A *be a set. A subset* R *of* $A \times A$ *is said to be an equivalence relation on* A *if the following conditions are satisfied:*

Reflexive Property *For each* $a \in A$, $(a, a) \in R$,

Symmetric Property *For all* $a, b \in A$, *if* $(a, b) \in R$, *then* $(b, a) \in R$, *and*

Transitive Property *For all* $a, b, c \in A$ *if* $(a, b) \in R$ *and* $(b, c) \in R$, *then* $(a, c) \in R$.

Problem 138 *Let* $A = \{a, b, c, d\}$. *Let*

$$R = \{(a, a), (a, c), (a, d), (c, a), (c, c), (c, d), (d, a), (d, c), (d, d)\}.$$

Is R *and equivalence relation on* A*?*

Problem 139 *Let* $A = \{a, b, c, d\}$. *Give an example of a subset of* $A \times A$ *which satisfies the reflexive and symmetric properties, but is not an equivalence relation.*

Problem 140 *Let* $A = \{a, b, c, d\}$. *Give an example of a subset of* $A \times A$ *which satisfies the symmetric and transitive properties, but is not an equivalence relation.*

Problem 141 *Let* $A = \{a, b, c, d\}$. *Give an example of a subset of* $A \times A$ *which satisfies the reflexive and transitive properties, but is not an equivalence relation.*

Problem 142 *Let* A *be a set. Let* $R = A \times B$. *Is* R *an equivalence relation on* A*?*

Chapter 7

Fraction Model for the Numbers, Parker

7.1 Introduction to the instructor

These notes are intended for a course in which students may be proving theorems on their own for the first time. One axiomatic development for the numbers takes *number*,$<$,$+$, and $*$ as primitive words, and establishes axioms that guarantee a field structure for $+$ and $*$ on the set of numbers; establishes $<$ to be an order on the set of numbers which is Dedekind complete, has no maximum nor minimum, and admits a denumerable subset of the numbers which is dense in it; and connects the algebra of $+$ and $*$ to the geometry of $<$. For students who are truly naïve mathematically, this gives little context for making examples since they may have only a vague recognition that the numbers they use as tools even have these properties. In these notes, the mathematical content addresses the question "Can we define objects, establish a comparison principle among them, and impose an algebra on them so that if the objects are interpreted as numbers, the comparison principle is used to define $<$, and the algebra used to define $+$ and $*$, then the statements for the axioms can be demonstrated to be true for these meanings?"

In these notes, a fraction model is pursued. The students are given that the natural numbers exist and questions of number theory may be deflected or pursued as the instructor chooses. Students are given the natural numbers and addition, multiplication, subtraction, and division on the natural numbers along with the admonition that, whereas addition and multiplication always make sense, $5 - 3$ is okay, but $3 - 5$ is not, and $9 \div 3$ makes sense, but $5 \div 3$ does not since neither $3 - 5$ nor $5 \div 3$ is a natural number. My general choice is to accept those statements the students make about number theory that are actually true as true. I typically grant the infinitude of the primes, unique prime factorization, and the Euclidean algorithm. However, in a recent semester, a student proved the Euclidean algorithm as a part of an argument; I was glad I had not given the class that theorem. Sometimes I ask for a proof, particularly if a proof would be likely to afford an opportunity to teach a proof technique (such as finite induction). Working within the fraction model gives fruitful opportunities for application of number theory theorems and students often make arguments about the number theory in the context of arguing about the model.

The problem set is designed so that even a class which plods will be exposed to ideas of comparing sets, imposing an order on a set, and imposing an algebra on a set. A class that experiences success from the beginning can be expected to get at least to the point of recognizing that the fraction model is not Dedekind complete. I teach these notes as a one-semester course. I have taught classes that extended the order to the completion, but I have never had a class get through the corresponding extension of the algebra to the completion.

The story line for the course goes like this:

- Defining a fraction consists of specifying the numerator and denominator.

- With the comparison principle from classical geometry, the product of the extremes is less than the product of the means, different objects may not be comparable.
- Having corrected this flaw, the set admits an order with no minimum and no maximum and with the property that between any two elements is another element.
- Using the algebra algorithms from grade school arithmetic and being careful, the set admits an order-preserving semigroup which lacks an identity element, and admits a multiplication which is a group and distributes over the semigroup.
- The set fails to have the Dedekind cut property for its comparison principle.
- Correct the flaw and make the necessary adjustments to the algebra.

Meanwhile, we are also counting sets.

- The natural numbers are shown to be infinite.
- Segments are shown to be as large as the entire set.
- The natural numbers are discovered to be as large as the (pre-Dedekind) set.
- The natural numbers are shown to be not as large as any Dedekind-complete set.

Instructors considering using these notes for a course should be aware of certain intentional idiosyncrasies built into them. I do not forbid my students to use books or any non-human sources they can find. I don't encourage it either — if they ask if it is okay, I just tell them to make sure it doesn't keep them from solving problems. I do consciously make the problem set so that I think it is simpler to work on the problems directly than to find an appropriate source and make the necessary translations. Thus some very standard problems may be cast in molds that are not immediately recognizable. For instance "the fraction with numerator 3 and denominator 4" is (3,4) in the model. If students don't realize this before encountering the density problem, sit back and enjoy how creative they get in building a replacement for the midpoint formula. The longer it takes, the more opportunities you will get to appreciate how rich the order structure for the rational numbers is when liberated from its algebra, because the students will think of the objects in ways those of us with too much education can't even imagine. Of course, once a student makes this connection, grade school arithmetic becomes an available source for intuition and more rapid progress will be made through the notes. However, I consider having the student make that connection her/himself inviolate, and I would neither confirm nor deny the connection if asked. My standard response would be, "What I do know is that the definitions mean what the definitions say." As another example, I use "x and y are elements" and "each of x and y is an element" to distinguish contexts that are typically explained in books using the word distinct. Besides being correct, (distinct may also be used correctly, even if it is often used redundantly) the usage allows me to teach lessons about language I consider important. I use the term "the numbers" where most sources use the term "the real numbers". This is not important; I just do not like to miss a chance to voice my own personal prejudice that the complex numbers are every bit as real as the "real numbers". I don't want to be personally responsible for perpetuating the adjective "imaginary" when it might be interpreted by the unlearned as being an opposite in some sense to "real."

Since progress through these notes depends on the students finding the conjectures which are not theorems and then addressing the issues raised in trying to produce structures about which the conclusions *are* true, the instructor must exercise some care in *when* access to subsequent pages is granted to the students. Also, students often uncover theorems while working on problems, and theorems can often be sifted out of students' arguments; these make nice addenda to the notes. I have included some examples from a past class of mine in Section 10.4. Keep in mind that these are merely statements of theorems that the students' proofs turned up; the students may not have stated the theorem that way and the theorem itself may not have solved the problem it addressed. Section 10.5 contains remarks about particular problems or definitions and possible timing schemes for presenting the problems. I have taught Mathematics 315 at James Madison University using some form of the notes to follow many times. In this section, I have tried to give the benefit of some of my experience without being too heavy-handed about what "ought" to happen. It is my hope that the remarks might help with the order in which the problems should be presented. The order I have given need not be the optimal sequence for a particular class.

I give only one test in this course, the final examination. I offer one credit each time a student presents an argument for a problem that the class judges as being correct. If a student has a problem that someone else presents, that student is allowed to turn in her/his write-up at the end of the class period in which the problem was finished. If the write-up is correct, the student gets one-half of a credit. To discourage note taking and to encourage the students to think back through what they have seen, I offer myself as a source for any problem that has been solved by the class. This way, if a student cannot reproduce an argument, I am a safety net for their being able to see the proof again, and if necessary, again, and ... The final examination is given as a take-home, use your notes but nobody else's, test. It consists of a section of problems solved during the semester and a section of problems that the students have either not yet solved or have not yet seen. Successful proofs for the problems proven during the course allow students to make C or to keep whatever grade their work during the semester warranted. Successful work on the second section allows a student to increase her/his grade or to atone for slip-ups on the first section.

7.2 Introduction to students - An Introduction to Doing Mathematics

In this course, not only will you be responsible for understanding why the mathematics we cover is correct, but the responsibility for discovery will also be assigned to the class. One of the immediate results of this responsibility for doing mathematics yourself rather than just learning how someone else did it will likely be an acute awareness of the difference between the challenge associated with understanding why something is correct and discovering for yourself whether or not a conjecture is a theorem.

Doing mathematics can be extremely exhilarating when one succeeds in the discovery process; failing to do mathematics when one is putting in the time trying to do mathematics can be extremely frustrating. This introduction is designed to alert you to some tips that are designed to optimize the chances for success.

First, you must put in the time necessary to give your creative intelligence a chance to work. Flashes of insight typically occur after information is organized and mulled over. Commitment to solving problems often leads to help from the subconscious. Students often tell me that they got "the big idea" while walking across campus or after turning in for the night.

Second, solutions to problems need not come all at once. You may need to solve many small problems on the way to proving a theorem or disproving an incorrect conjecture. Some of the most important work in mathematics is the creation of technique. Take pride in progress toward a goal as well as reaching the goal. Any information you uncover is more than you knew before, and solving a problem is usually just a matter of putting together enough small solutions to allow you to see why the big problem is correct.

Students often tell me that they would be glad to put in the time if they just knew where to start. The following scheme is offered toward that end.

The awareness stage

1. Identify all the words in the problem and make sure that you *know* the definition of each of them. Try to recall examples that have dealt with these notions before. If a definition is new, make some examples for the definition.

2. Identify any theorems that may have already dealt with ideas present in the problem. Put techniques that gave rise to proofs in those contexts firmly in mind.

The direct approach

3. Make an example that models the hypothesis to the problem and try to show that the example exhibits the properties of the conclusion. (If you can prove that your example fails to have the properties of the conclusion, you will have shown that the problem is not a theorem!)

4. See if what allowed you to establish the conclusion in the example is a property of all examples covered by the hypothesis. If it is, write a proof. If not,

5. ...make an example which models the hypothesis but fails to have whatever special properties you used to get the conclusion in the previous example. Go to 3.

The indirect, or contrapositive approach

6. Suppose that the conclusion is false and try to show that the hypothesis must be false as well. If the problem is not a theorem, any conclusions you get must be qualities an example that disproves the conjecture must have.

7. Try to be aware of properties that, if they were added to the hypothesis, would guarantee the conclusion. Alternatively, you might also try to find conclusions that follow from the hypothesis, even if they do not include the one you seek. Even if you are not able to solve the problem as stated, you may be able to create a substitute theorem.

The main mindset is to be aware that even when arguments do not come quickly or easily, the hunt itself may be an important learning experience. Working on problems yourself is the central ingredient. Not only will it provide you with theorems that are "your own," but even when someone beats you to a solution, it will put you in a much stronger position to analyze the argument given.

A theory of sets and ordered pairs

We will not create an axiomatic set theory. Following, however, is an idiomatic presentation of some conventions that axiomatic set theory implies. We presuppose the existence of formal English as a language for expressing properties.

The primitive words are set, element, ordered pair, first co-ordinate, and second co-ordinate.

i. A set consists of an element or elements.

ii. An element of a set and the set consisting of that element are different objects.

iii. A set is defined by stating the properties its elements have. (The plural has been chosen here, but the definition of a set may be made by stating a single property and a set may have a single element.)

iv. Given a definition for a set, any object having the properties specified is an element of the set; and any element of the set has the properties specified in the definition.

v. An ordered pair consists of a first co-ordinate and a second co-ordinate.

vi. The first co-ordinate of an ordered pair may be the same set-theoretic object as the second co-ordinate, but as a part of the ordered pair, being the first co-ordinate is distinguishable from being the second co-ordinate.

We reserve a notation for the creation of definitions of sets and for defining ordered pairs.

Reserved symbols for definitions of sets are { : }. A symbol is created to follow the open brace and precede the colon and then properties that an element must have are stated in terms of that symbol after the colon and before the closed brace. Thus

$$\{x : x \text{ is a number and } x > 5\}$$

stands for "the set to which an element belongs provided that it is a number and it is greater than 5."

Reserved symbols for definitions of ordered pairs are (,). The first co-ordinate of the ordered pair is written after the open parenthesis and before the comma; the second co-ordinate of the ordered pair is written after the comma and before the closed parenthesis. Thus $(p, 5)$ stands for the ordered pair whose first co-ordinate is p and whose second co-ordinate is 5.

The purpose of this course is to build a model for the numbers. Our ultimate goal is to prove that the statements which are typically taken as axioms for the numbers are theorems in our model. In an axiomatic treatment, *number*, $<$, $+$, and $*$ are taken as primitive words; thus we provide definitions within the model so that if they are interpreted as the primitive words, the statements made by replacing the primitive words in the axioms with their analogues in the model become the topics of consideration.

You may assume that the natural numbers exist and have whatever properties number theory says they do. If there is doubt about a property of the natural numbers, we will either prove the property or indicate what property we are assuming.

7.3 Problem Sequence

Definition 1 Suppose that each of X and Y is a set. The statement that f is a **function from X into Y** means that f is a set so that

i. each element of f is an ordered pair whose first co-ordinate is an element of X and whose second co-ordinate is an element of Y; and

ii. if p is an element of X, then there is an element of f whose first co-ordinate is p; and

iii. if p and q are elements of f, then the first co-ordinate of p is not the first co-ordinate of q.

Notation: If f is a function from X into Y and (p,q) is an element of f, then we may write f(p) = q.

Definition 2 Suppose that each of X and Y is a set and that f is a function from X into Y. The statement that M is the **range of f** means that M is the set to which an element belongs provided that there is an element of f of which it is the second co-ordinate.

Problem 1 Suppose that X is a set with more than one element[1]. Show that the set to which an element belongs provided that it is an ordered pair whose first co-ordinate is an element of X and whose second co-ordinate is an element of X is not a function from X into X.

Definition 3 Suppose that X is a set and that L is a set each element of which is an ordered pair whose first co-ordinate is an element of X and whose second co-ordinate is an element of X. The statement that L is an **order on X** means that

i. if p is an element of X, then (p,p) is not an element of L; and

ii. if p and q are elements of X, then (p,q) is an element of L or (q,p) is an element of L; and

iii. if (p,q) and (q,r) are elements of L, then (p,r) is an element of L.

Problem 2 Suppose that X is a set with exactly one element. Show that there is no order on X.

Problem 3 Suppose that X is a set with more than one element and that L is an order on X. Show that L is not a function from X into X.

Definition 4 U is the set to which an element belongs provided that it is an ordered pair each of whose co-ordinates is a natural number.

[1] That X has more than one element means that if p is an element of X, then there is an element of X different from p.

Definition 5 Suppose that (a,b) and (x,y) are elements of U. The statement that (a,b) **precedes** (x,y) means that a $*$ y is less than x $*$ b.

Definition 6 G = {(p,q) : p is an element of U, and q is an element of U, and p precedes q}.

Problem 4 Suppose that x is an element of U. Show that there is an element of U, call such an element y, so that (x,y) is an element of G.

Problem 5 Suppose that x is an element of U. Show that there is an element of U, call such an element y, so that (y,x) is an element of G.

Problem 6 Suppose that x and y are elements of U and (x,y) is an element of G. Show that there is an element of U, call such an element w, so that (x,w) and (w,y) are elements of G.

Problem 7 Show that G is an order on U.

Definition 7 Suppose that each of X and Y is a set. The statement that **X commands Y** means that there is a function from X into Y whose range is Y.

Problem 8 Show that U commands the natural numbers.

Definition 8 Suppose that each of X and Y is a set. The statement that **X is a subset of Y** means that if p is an element of X, then p is an element of Y.

Problem 9 Suppose that each of X and Y is a set and that X is a subset of Y. Show that Y commands X.

Problem 10 Suppose that X and Y are sets and that X is a subset of Y. Show that it is not the case that X commands Y.

Definition 9 Suppose that each of X and Y is a set and that there is an element of X which is an element of Y. The **intersection of X with Y** is {x : x is an element of X and x is an element of Y}.

Notation: X $\cap$ Y stands for "the intersection of X with Y".

Definition 10 Suppose that X is a set, L is an order on X, and a and b are elements of X so that (a,b) is an element of L, and there is an element of X, call such an element c, so that (a,c) is an element of L and (c,b) is an element of L. The **segment from a to b by L** is {x : (a,x) is an element of L and (x,b) is an element of L}.

Notation: If (a,b) is an element of the order L, then $\underline{(}a,b\underline{)}$ stands for "the segment from a to b by L".

Problem 11 Suppose that x is an element of U. Show that there is a segment by G so that x is an element of it.

Problem 12 Suppose that X is a set, L is an order on X, $\underline{(}p,q\underline{)}$ and $\underline{(}r,s\underline{)}$ are segments by L, and x is an element of $\underline{(}p,q\underline{)} \cap \underline{(}r,s\underline{)}$. Show that $\underline{(}p,q\underline{)} \cap \underline{(}r,s\underline{)}$ is a segment by L.

Problem 13 Suppose that x and y are elements of U and that G is an order on U. Show that there are segments by G, call them P and Q, so that

i. x is an element of P,

ii. y is an element of Q, and

iii. if w is an element of P, then w is not an element of Q.

Definition 12 Suppose that each of X and Y is a set. The **union of X with Y** is {p : p is an element of X or p is an element of Y}.

Notation: X ∪ Y stands for "the union of X with Y".

Definition 13 Suppose that X is a set, L is an order on X, and T and V are subsets of X. The statement that (T,V) is a **cut of X by L** means that

i. T ∪ V = X; and

ii. if x is an element of T and y is an element of V, then (x,y) is an element of L.

Problem 14 Suppose that X is a set, L is an order on X, and (A,B) is a cut of X by L. Show that if p is an element of A, then p is not an element of B.

Problem 15 Suppose that X is a set and L is an order on X. Show that there is a cut of X by L.

Problem 16 Suppose that X is a set, L is an order on X, and (p,q) is an element of L. Show that there is a cut of X by L, call it (A,B), so that p is an element of A and q is an element of B.

Definitions 14 Suppose that X is a set, L is an order on X, p is an element of X, and M is a subset of X. The statement that p is the **max of M by L** means that p is an element of M, and if q is an element of M different than p, then (q,p) is an element of L. The statement that p is the **min of M by L** means that p is an element of M, and if q is an element of M different than p, then (p,q) is an element of L.

Definition 15 Suppose that X is a set and L is an order on X. The statement that L has the **Dedekind cut property** means that if (A,B) is a cut of X by L, then

i. A has a max by L or B has a min by L and

ii. it is not the case that both A has a max by L and B has a min by L.

Problem 17 Suppose that L = {(x,y) : x is a natural number, y is a natural number, and $x < y$}. Show that L does not have the Dedekind cut property.

Definition 4' U' = {(x,y) : (x,y) is an element of U; and if each of a and b is a natural number so that $a > 1$ and $a * b = x$, and each of c and d is a natural number so that $y = c * d$, then a is not c and a is not d}.

Definition 6' G' = {(p,q) : p is an element of U', q is an element of U', and p precedes q}.

Problem 4' Suppose that x is an element of U'. Show that there is an element of U', call such an element y, so that (x,y) is an element of G'.

Problem 5' Suppose that x is an element of U'. Show that there is an element of U', call such an element y, so that (y,x) is an element of G'.

Problem 6' Suppose that x and y are elements of U' and (x,y) is an element of G'. Show that there is an element of U', call such an element w, so that (x,w) and (w,y) are elements of G'.

Problem 7' Show that G' is an order on U'.

Problem 8' Show that U' commands the natural numbers.

Problem 11' Suppose that G' is an order on U' and that x is an element of U'. Show that there is a segment by G' so that x is an element of it.

Problem 18 Suppose that
T = {(p,q) : (p,q) is an element of U' and $p * p < 2 * q * q$} and
V = {x : x is an element of U' and x is not an element of T}.
Show that (T,V) is a cut of U' by G'.

Problem 19 Suppose that
T = {(p,q) : (p,q) is an element of G' and $p * p < 2 * q * q$} and
V = {x : x is an element of G' and x is not an element of T}.

Possible conclusion i: T has a max by G'.

Possible conclusion ii: V has a min by G'.

Possible conclusion iii: T does not have a max by G' and V does not have a min by G'.

Problem 20 Show that

Possible conclusion i: it is not the case that U' commands U.

Possible conclusion ii: U' commands U.

Problem 21 Show that

Possible conclusion i: it is not the case that **N** commands U'.

Possible conclusion ii: **N** commands U'.

Problem 22 Suppose that (x,y) is an element of G'.

Possible conclusion i: Show that it is not the case that $\underline{(}x,y\underline{)}$ commands U'.

Possible conclusion ii: Show that $\underline{(}x,y\underline{)}$ commands U'.

Definition 16 ρ = {((x,y),(p,q)),z) : (x,y) is an element of U', (p,q) is an element of U', and z = ((x ∗ q) + (p ∗ y),y ∗ q)}

Problem 23 Show that ρ is a function from {(x,y) : x is an element of U' and y is an element of U'} into U'.

Definition 17 τ = {((x,y),(p,q)),z) : (x,y) is an element of U', (p,q) is an element of U', and z = (x ∗ p,y ∗ q)}

Problem 24 Show that τ is a function from {(x,y) : x is an element of U' and y is an element of U'} into U'.

Definition 18 Suppose that (x,y) is an element of U'. $EC_{(x,y)}$ = {(p,q) : there is a natural number, call it k, so that p = k ∗ x and q = k ∗ y}.

Problem 25 Suppose that (x,y) is an element of U. Show that there is an element of U', call it w, so that (x,y) is an element of EC_w.

Problem 26 Suppose that (p,q) and (x,y) are elements of U' and that (r,s) is an element of $EC_{(p,q)}$. Show that (r,s) is not an element of $EC_{(x,y)}$.

Problem 27 Suppose that (p,q) and (x,y) are elements of U' so that ((p,q),(x,y)) is an element of G', and (r,s) is an element of $EC_{(p,q)}$ and (w,z) is an element of $EC_{(x,y)}$. Show that ((r,s),(w,z)) is an element of G.

Definition 16' ρ' = {((x,y),(p,q)),z) : (x,y) is an element of U', and (p,q) is an element of U', and z is an element of U', and ((x ∗ q) + (p ∗ y),y ∗ q) is an element of EC_z}

Problem 23' Show that ρ' is a function from {(x,y) : x is an element of U' and y is an element of U'} into U'.

Definition 17' τ' = {((x,y),(p,q)),z) : (x,y) is an element of U', and (p,q) is an element of U', and z is an element of U', and (x ∗ p,y ∗ q) is an element of EC_z}

Problem 24' Show that τ' is a function from {(x,y) : x is an element of U' and y is an element of U'} into U'.

Problem 28 Suppose that x and y are elements of U'.
Show that ρ'((x,y)) = ρ'((y,x)).

Problem 29 Suppose that x and y are elements of U'.
Show that τ'((x,y)) = τ'((y,x)).

Problem 30 Suppose that each of x, y, and z is an element of U'.
Show that ρ'((ρ'((x,y)),z)) = ρ'((x, ρ'((y,z)))).

Problem 31 Suppose that each of x, y, and z is an element of U'.
Show that τ'((τ'((x,y)),z)) = τ'((x,τ'((y,z)))).

Problem 32 Suppose that each of x, y, and z is an element of U'.
Show that τ'((x,ρ'((y,z)))) = ρ'((τ'((x,y)), τ'((x,z)))).

Problem 33 Show that there is an element of U', call such an element ζ, so that if x is an element of U', then ρ'((x,ζ)) = x and ρ'((ζ,x)) = x.

Problem 34 Show that there is an element of U', call such an element μ, so that if x is an element of U', then τ'((x, μ)) = x and τ'((μ,x)) = x .

Problem 35 Suppose that x and y are elements of U', (x,y) is an element of G', and z is an element of U'. Show that (ρ'((x,z)), ρ'((y,z))) is an element of G'.

Problem 36 Suppose that x and y are elements of U' so that (x,y) is an element of G'. Show that there is exactly one element of U', call it q, so that ρ'((x,q)) = y.

Problem 37 Suppose that x is an element of U'. Show that there is exactly one element of U', call it y, so that τ'((x,y)) = (1,1).

Definition 4" U" = {x : x is an element of U'; or
there is a cut of U' by G', call it (A,B), so that
A has no maximum by G' and
B has no minimum by G', and x = A}.

Definition 6" G" = {(x,y) : x and y are elements of U"; and
if x and y are elements of U', then (x,y) is an element of G', or
if x is an element of U' and y is not an element of U',
then x is an element of y, or

if x is not an element of U' and y is an element of U',
then y is not an element of x, or
if neither x nor y is an element of U', then x is a subset of y}

Problem 4" Suppose that x is an element of U". Show that there is an element of U", call such an element y, so that (x,y) is an element of G".

Problem 5" Suppose that x is an element of U". Show that there is an element of U", call such an element y, so that (y,x) is an element of G".

Problem 6" Suppose that x and y are elements of U" and (x,y) is an element of G". Show that there is an element of U', call such an element w, so that (x,w) and (w,y) are elements of G".

Problem 7" Show that G" is an order on U".

Problem 38 Show that G" has the Dedekind cut property.

Problem 22" Suppose that (x,y) is an element of G". Show that (x,y) commands U".

Definition 19 Suppose that X is a set, L is an order on X, and (p,q) is an element of L. The **interval from p to q by L** is {x : x is an element of (p,q), or x is p,or x is q}.

Notation: If (p,q) is an element of the order L, [p,q] stands for "the interval from p to q by L".

Definition 20 Suppose that X is a set. The statement that **s is a sequence in X** means that s is a function from the natural numbers into X.

Problem 39 Suppose that M = {x : there is an element of G", call it (p,q), so that x = [p,q]}. Show that there is a sequence in M, call such a sequence f, so that if n is a natural number then, then f(n+1) is a subset of f(n).

Problem 40 Suppose that (A, B) is an element of G" and x is an element of U" so that x is an element of (A,B). Show that there is an element of G", call it (p,q), so that [p,q] is a subset of (A,B) and x is not an element of [p,q].

Problem 41 Suppose that

M = {x : there is an element of G”, call it (p,q), so that x = [p,q]},
s is a sequence in M so that if k is a natural number, then
s(k+1) is a subset of s(k), and
A = {x : there is a natural number, call it k, so that
if p is an element of s(k), then (x,p) is an element of G”}.
Show that (A,{x : x is an element of U” and x is not an element of A})
is a cut of U” by G”.

Problem 42 Suppose that
M = {x : there is an element of G”, call it (p,q), so that x = [p,q]},
s is a sequence in M so that if k is a natural number, then
s(k+1) is a subset of s(k). Show that there is an element of U”, call it w, so that if k is a natural number, then w is an element of s(k).

Problem 43 Show that the natural numbers do not command U”.

Problem 44 Suppose that (A,B) is a cut of U’ by G’ so that A has no max by G’, p is an element of U’, and C = {x : x is an element of A, or there is an element of A, call it q, so that x = ρ’((p,q))}. Show that C has no max by G’.

Problem 45 Suppose that (A,B) is a cut of U’ by G’ so that A has no max by G’, p is an element of U’, and C = {x : x is an element of A, or there is an element of A, call it q, so that x = ρ’((p,q))}. Show that
(C,{x : x is an element of U’ and x is not an element of C}) is a cut of U’ by G’.

Problem 46 Suppose that (A,B) is a cut of U’ by G’ so that A has no max by G’, that (D,E) is a cut of U’ by G’ so that D has no max by G’, and that C = {x : x is an element of A, or there is an element of A, call it p, and an element of D, call it q, so that x = ρ’((p,q))}. Show that C has no max by G’.

Problem 47 Suppose that (A,B) is a cut of U’ by G’ so that A has no max by G’, that (D,E) is a cut of U’ by G’ so that D has no max by G’, and that C = {x : x is an element of A, or there is an element of A, call it p, and an element of D, call it q, so that x = ρ’((p,q))}. Show that
(C,{x : x is an element of U’ and x is not an element of C}) is a cut of U’ by G’.

Problem 48 Suppose that (A,B) is a cut of U’ by G’ so that A has no max by G’, p is an element of U’, and C = {x : there is an element of A, call it q, so that x = τ’((p,q))}. Show that C has no max by G’.

Problem 49 Suppose that (A,B) is a cut of U’ by G’ so that A has no max by

G', p is an element of U', and C = {x : there is an element of A, call it q, so that x =τ'((p,q))}. Show that
(C,{x : x is an element of U' and x is not an element of C}) is a cut of U' by G'.

Problem 50 Suppose that (A,B) is a cut of U' by G' so that A has no max by G', that (D,E) is a cut of U' by G' so that D has no max by G', and that
C = {x : there is an element of A, call it p, and an element of D, call it q,
so that x =τ'((p,q))}. Show that C has no max by G'.

Problem 51 Suppose that (A,B) is a cut of U' by G' so that A has no max by G', that (D,E) is a cut of U' by G' so that D has no max by G', and that
C = {x : there is an element of A, call it p, and an element of D, call it q, so that
x =τ'((p,q))}. Show that
(C,{x : x is an element of U' and x is not an element of C}) is a cut of U' by G'.

Definition 16" ρ" = {((x,y),z) : x is an element of U" and y is an element of U"; and if each of x and y is an element of U', then z = ρ'((x,y)), or
if x is an element of U' and y is not an element of U', then
z = {w : w is an element of y, or there is an element of y, call it q, so that w = ρ'((x,q))}}, or
if x is not an element of U' and y is an element of U', then
z = {w : w is an element of x, or there is an element of x, call it q, so that w = ρ'((q,y))}}, or
if x is not an element of U' and y is not an element of U', then
z = {w : w is an element of x; or there is an element of x, call it p, and there is an element of y, call it q, so that w = ρ'((p,q))}}.

Problem 23" Show that ρ" is a function from {(x,y) : x is an element of U" and y is an element of U"} into U".

Definition 17" τ" = {((x,y),z) : x is an element of U" and y is an element of U"; and if each of x and y is an element of U', then
z = τ'((x,y)), or
if x is an element of U' and y is not an element of U', then
z = {w : there is an element of y, call it q, so that w = τ'((x,q))}, or
if x is not an element of U' and y is an element of U', then
z = {w : there is an element of x, call it q, so that w = τ'((q,y))}, or
if x is not an element of U' and y is not an element of U', then
z = {w : there is an element of x, call it p, and there is an element of y, call it q, so that w = τ'((p,q))}}.

Problem 24" Show that τ" is a function from {(x,y) : x is an element of U" and y is an element of U"} into U".

Problem 28" Suppose that x and y are elements of U".
Show that ρ"((x,y)) = ρ"((y,x)).

Problem 29" Suppose that x and y are elements of U".
Show that τ"((x,y)) = τ"((y,x)).

Problem 30" Suppose that each of x, y, and z is an element of U".
Show that ρ"((ρ"((x,y)),z)) = ρ"((x,ρ"((y,z)))).

Problem 31" Suppose that each of x, y, and z is an element of U".
Show that τ"((τ"((x,y)),z)) = τ"((x,τ"((y,z)))).

Problem 32" Suppose that each of x, y, and z is an element of U".
Show that τ"((x,ρ"((y,z)))) = ρ"((τ"((x,y)), τ"((x,z)))).

Problem 35" Suppose that x and y are elements of U", (x,y) is an element of G", and z is an element of U". Show that (ρ"((x,z)), ρ"((y,z))) is an element of G".

Problem 42" Suppose that x and y are elements of U" so that (x,y) is an element of G". Show that there is exactly one element of U", call it q, so that ρ"((x,q)) = y.

Problem 43" Suppose that x is an element of U". Show that there is exactly one element of U", call it y, so that τ"((x,y)) = (1,1).

7.4 Examples

Following are some theorems that students proved during my Spring 1999 offering of the course for which a selection from the problems above formed the corpus from which they worked. The problem from the notes which each addresses is noted in parentheses. The theorem stated may not solve that problem; the reference is intended to show what students might be thinking when working on the problem. The statements of the theorems typically include the structure the student created to address the referenced problem.

Theorem(6) Suppose that (a,b) and (c,d) are elements of U and (a,b) precedes (c,d). Then ((a $*$ d)+(c $*$ b),2 $*$ b $*$ d) is an element of U so that (a,b) precedes ((a $*$ d)+(c $*$ b),2 $*$ b $*$ d), and ((a $*$ d)+(c $*$ b),2 $*$ b $*$ d) precedes (c,d). (This is a nice example of using the arithmetic of the "midpoint" to find a candidate for the solution. It is also nice in that the element produced need not be in U' even when both (a,b) and (c,d) are, so there is still something to be discovered if a student were to try to use the construction that worked here on Problem 6'.)

Theorem(6') Suppose that (a,b) and (c,d) are elements of U' so that (a,b) precedes (c,d) and so that a $<$ b, d $<$ c, (a,d) is an element of U', and (c,b) is an element of U'. Then (a $*$ c, b $*$ d) is an element of U' so that (a,b) precedes (a $*$ c, b $*$ d) and (a $*$ c, b $*$ d) precedes (c,d). (Here is an example of a theorem that only partially solves Problem 6'. It is not unusual for students to case such problems before finding a solution such as the one above that handles all cases at once.)

Definition: Suppose that each of n and m is a natural number. k is the **greatest common factor of n and m** means that there is a natural number, call it p, so that n = k $*$ p and there is a natural number, call it q, so that m = k $*$ q; and (p,q) is an element of U'. (The student here made a definition for a term with which he had pre-existing familiarity in terms of the structure of the course!)

Notation: gcd{a,b} stands for "the greatest common factor of a and b."

Theorem(25) Suppose that (a,b) is an element of U.
Then $\left(\frac{a}{gcd\{a,b\}}, \frac{b}{gcd\{a,b\}}\right)$ is an element of U'.

Theorem(30) Suppose that each of (a,b), (c,d), and (e,f) is an element of U'. Then

$$EC_{(a*((c*f)+(d*e)),b*(d*f))} = EC_{(a*\frac{(c*f)+(d*e)}{gcd\{(c*f)+(d*e),d*f\}},b*\frac{d*f}{gcd\{(c*f)+(d*e),d*f\}})}.$$

Theorem(21) Define **a** by **a**(1) = (3,2) and if n is a natural number, then

$$\mathbf{a}(n+1) = \left(\frac{\Pi_1\mathbf{a}(n) + 2 * \Pi_2\mathbf{a}(n)}{gcd\{\Pi_1\mathbf{a}(n) + 2 * \Pi_2\mathbf{a}(n), \Pi_1\mathbf{a}(n) + \Pi_2\mathbf{a}(n)\}}, \frac{\Pi_1\mathbf{a}(n) + \Pi_2\mathbf{a}(n)}{gcd\{\Pi_1\mathbf{a}(n) + 2 * \Pi_2\mathbf{a}(n), \Pi_1\mathbf{a}(n) + \Pi_2\mathbf{a}(n)\}}\right).$$

Then **a** is a function from the natural numbers into U'.

Theorem(19) Suppose that
A = {(p,q) : (p,q) is an element of U' and $p * p < 2 * q * q$} and
B = {x : x is an element of U' and x is not an element of A}.
Then B = { (p,q) : (p,q) is an element of U' and $p * p > 2 * q * q$}, and
if (a,b) is an element of U', then

$$\left(\frac{(3*a)+(4*b)}{gcd\{(3*a)+(4*b),(2*a)+(3*b)\}},\frac{(2*a)+(3*b)}{gcd\{(3*a)+(4*b),(2*a)+(3*b)\}}\right)$$

is an element of U' so that if (a,b) is an element of A, then
it is an element of A and (a,b) precedes it, or
if (a,b) is an element of B, then
it is an element of B and it precedes (a,b).

7.5 Remarks

The following comments contain information about my intent for many of the problems and experiences that my students have had with them.

Definition 1 Notice that the idea of domain is included in the definition of function by proviso ii..

Definition 2, Problem 1, Definition 4 I have written the words for definitions of the sets in question here. Students often translate these to the notation for these words suggested back on the page on sets. If they don't, I usually suggest that they see if they can.

Problems 1 & 3 The students I teach in this course are typically very, very naïve and seldom have been forced to deal with the need to say things carefully. I put Problems 1 & 3 in the notes to address the fact that early in every course, a student would define a "function" as {(x,y) : x is an element of X and y is an element of Y} and then claim whatever additional properties he or she needed as he or she needed them. When it happens now, I can point to Theorem 1 and start the discussion there. Problem 3 reinforces "not every set of ordered pairs whose co-ordinates are in the right sets is a function."

Problem 2 This problem is here to emphasize that, in this set theory, sets have at least one element each. Discussion of the "empty set" will naturally occur here.

Definition 6 Notice that p and q represent ordered pairs here. This forces the students to take meaning from the words and, when arguing from or to Definition 6, gives an opportunity for them to rewrite their instantiations so that the co-ordinates are explicitly denoted.

Problems 4, 5, & 6 These are all properties assumed about $<$ on $\Re$. Although U will be modified when it fails to support precedes as an order, the proofs made in U for 4, 5, & 6, properly adjusted, usually go over to all subsequent modifications.

Problem 7 This is the first false conjecture since, for instance, (2,3) and (4,6) cannot be compared by precedes. Historically, the resolution to this problem was to identify all such objects as representing the same thing, an idea formalized with equivalence classes. I prefer to emphasize that, in a model, different objects must be different and to continue with a subset of the objects with which we are working. (In "real life", i.e., the world of computation, these are the only objects which are accessible. Equivalence classes come up again and again anyway, so the idea is still there to be studied later.)

Definition 7 Commands is the concept central to counting. The Schröder-Bernstein Theorem, which is not addressed in this course, guarantees that it is sufficient to admit the classical results.

Problems 9 & 10 Although all students in your class will "know" that tangent maps $(-\pi/2,\pi/2)$ onto $\Re$, they will not likely realize that this precludes Problem 10 from being a theorem. Indeed, most of the time classes try to prove Problem 10 and the argument typically includes as a punch line something equivalent to "because the containing set contains more elements than the subset." This affords a marvelous opportunity to teach the difference between ordinary language and formal language since "more" in the subset sense turns out to be different than "more" in the counting sense (sometimes!). The questions that show incorrect arguments are incorrect often lead to the example $n+1 \rightarrow n$.

Problem 11 If Problem 7 has been done at this time, use Problem 11' here instead.

Problem 12 This problem virtually guarantees a case argument will be forthcoming.

Problem 13 Since Problem 7 is not a theorem, if this problem is solved, then it will be done using the order structure. Thus, together with Problem 12, we get that segments defined by an order with neither max nor min create a basis for a Hausdorff topology on the set on which the order is defined.

Definitions 13, 14, and 15 In an axiom system for the numbers, one has (at least) the choice of the greatest lower bound property, the Bolzano-Weierstrass property, the Heine-Borel property, and the Dedekind cut property as a completeness axiom. I choose the Dedekind cut property since it can be articulated without reference to any structure other than the order itself. This is the first idea in the course that the students are likely not to have encountered in another context.

Problem 14 An idea for proving Problem 14 has usually already been addressed by this point in the course. Usually students will argue as if they have "(a,b) or (b,a) and not both" instead of "(a,b) or (b,a)" in the definition of order, and can be forced to address whether or not this is a consequence of the definition at that time.

Problems 15 and 16 These problems are usually solved using cuts exhibiting the Dedekind cut property, thus establishing a pretext for asking "must all cuts be like these?"

Problem 17 The order, $<$, as we find it in counting, does not have the Dedekind cut property.

Definitions 4' and 6' and Problems 4'- 8' At some point, Problem 7 will be solved. The example that shows Problem 7 is not a theorem will show two elements, neither of which precedes the other. Often students demonstrate that properties i. and iii. from Definition 3 hold before providing the example that shows that property ii. fails. Sometimes a single example such as (1,1) and (2,2) is offered; sometimes students describe the phenomenon that is in Definition 4'. The instructor needs to make the structure in Definition 4' plausible from whatever platform the students provide. This may necessitate stating more problems, or initiating a class discussion in which other examples are created and the structure identified in Definition 4' is shown to be common to all of them. I have placed these problems after Problem 17 *only* because it has been typical in my experience that Problem 7 gets solved before Problems 15-17 get solved. Whenever Problem 7 gets solved, it is time for Definitions 4' and 6' and Problems 4' - 8'. Until Problem 7 gets solved, it is not time for Definitions 4' and 6' or Problems 4' - 8'. If students have discovered the "reduction argument" within equivalence classes, all five (six) problems will be direct consequences of it using the arguments from 4-8 (and 11) to get the objects from which to reduce. If students have not discovered the "reduction argument," an interesting sidelight is to see which constructions from 4-8 give elements of U'. These problems offer an outstanding context to point out the power of "some arguments" and always afford at least an opportunity to show how to modify an argument to meet new conditions.

Problems 18 & 19 Here is an example that shows that G' fails to have the Dedekind cut property.

Problems 20-22 Once Problem 10 gets solved and the possibility of seemingly "smaller" sets commanding seemingly "larger" sets is established, these problems are ready to be addressed. If U' has not been defined yet, state Problems 21 and 22 using U instead of U' and save " U' commands U " and the appropriate modifications of 21 and 22 for when U' is defined. Also, the problem can be stated with its correct conclusion to direct students or with the incorrect conclusion to misdirect students; I choose this particular presentation in order to illustrate that sometimes there is nothing in previous experience to indicate what the appropriate conjecture is.

Definitions 16 & 17, Problems 23 & 24 These are the algorithms for adding and multiplying fractions, but neither problem is a theorem. Both ρ and τ, however, are functions from U'xU' into U. In my experience, whenever I state algebra problems, the majority of the class becomes enamored of them and work on little else without severe prodding. I typically postpone stating them as late as I can. These are nice in that the counterexamples can be used to suggest that, even though the "answers" are different, they are "related" to the same element of U' and thus yet another opportunity is afforded for discovering the pertinent equivalence classes.

Definition 18, Problems 25-27 This definition and these problems are appropriate whenever the idea becomes clarified. Sometimes it happens when Problem 7 is settled; usually it happens by the time the U and G theorems are reproven for U' and G'. If it hasn't happened at this stage, they need to be stated since they are important for the algebra problems.

Problems 23', 24', and 28-34 These problems investigate the status of the field axioms. Problems 23' and 24' are closure properties, 28 and 29 are commutative properties, 30 and 31 are associative properties, 32 is the distributive property, and 33 and 34 are identity properties. Of these 33 is the only one which is not a theorem; indeed, if 35 were to be proven before 33, it shows that 33 cannot be true.

Problem 35 This is the property that addition preserves order.

Problem 36 & 37 Every equation for addition that can have a solution does have one and the reciprocal property holds for multiplication. 36 is appropriate once 33 is shown not to be a theorem and 35 is proven. 37 is appropriate once 34 is proven. If a class were to get this far, it would have established a model for the positive rational numbers and its algebra.

Definition 4" Be aware that Definition 4" introduces a new type of element to the model. Each element of U' is still there "... x is an element of U', or ...", but there are also elements which are subsets of U' "there is a cut of U' by G', call it (A,B), so that A has no maximum by G' and B has no minimum by G', and x = A." Dedekind made each number one of these (all you need do is drop the "A has no maximum and B has no minimum); I prefer the spirit of computation. These numbers are there, but our computational access to them is through our access to U', a denumerable set dense in U" by G". This is analogous to the situation in the decimal model where even though infinite decimal expansions exist, our computational access to them is typically through the arithmetic of numbers with terminating decimal expansions.

Definition 6" G" on U" models $<$ on the numbers.

Problems 4", 5", 6", 7", 21, & 38 These problems demonstrate that G" on U" models $<$ on the numbers. Note that in 6" the problem asks for an element of U', which, together with 21, gives a countable dense set. Geometrically, the structure may as well be the numbers. The lack of an additive identity for the algebra indicates that the model is more suitable for $<$ on the positive numbers.

Problems 39-43 Since any order with the Dedekind cut property has the "betweeness" property from which 40 is typically proven, these problems give a template for a proof that any order with the Dedekind cut property must be uncountable.

Problems 44-51 Prepare to extend the addition and multiplication to U".

Problems 23" and 24" The structure for extending the algebra is done in 44-51, but the student still must prove that the complement of "C" has no min.

Problems 23", 24", 28"-32", 35", 37" and 38" These establish that τ" and ρ" have the structure that is assumed for $*$ and $+$ on the positive numbers. If one desires to complete U", G", τ", and ρ" to a model for the numbers, $<$, $*$, and $+$; all that is needed is to introduce an object to be 0, and, for each element of U", an object that can be identified with it (I typically reserve the symbol $\tilde{x}$ for the object to be identified with x). The order is extended by letting 0 precede every element of U", every element of $\tilde{U}$" precede 0, and letting elements of $\tilde{U}$" reverse the order of the elements from U" from which they are built. τ" is extended by using the results on U" and supplying a rule of "signs"; τ"((x,y)) is in $\tilde{U}$" means exactly one or x and y is in $\tilde{U}$" and the object for the answer comes from looking at τ" as applied to the objects in U" from which x and/or y are built. Problems associated with these ideas are likely, for a class that has gotten this far, to be routine. I tend to use problems associated with these ideas as exam questions. Extending ρ" is more interesting, since it involves using solutions to equations on U" (those addressed in 42") for defining sums.

7.6 Some Axioms for the Numbers

The primitive words are number, $<$, $+$, and $*$.

Axiom G1 $<$ is an order on the set of numbers.

Axiom G2 It is not the case that the set of numbers has a min by $<$ and it is not the case that the set of numbers has a max by $<$.

Axiom G3 There is a sequence in the set of numbers, call it Q, so that if x and y are numbers, then there is a natural number, call it k, so that $x < Q(k)$ and $Q(k) < y$.

Axiom G4 $<$ has the Dedekind cut property.

Axiom A1 If each of x and y is a number, then x+y is exactly one number, and $x * y$ is exactly one number.

Axiom A2 If each of x and y is a number, then $x+y = y+x$, and $x * y = y * x$.

Axiom A3 If each of x, y, and z is a number, then $x+(y+z) = (x+y)+z$, and $x * (y * z) = (x * y) * z$.

Axiom A4 0 is a number so that if x is a number, then $0+x = x$, and 1 is a number so that if x is a number, then $1 * x = x$.

Axiom A5 If x is a number, then there is exactly one number, call it y, so that $x+y = 0$; and if x is a number different than 0, then there is exactly one number, call it w, so that $x * w = 1$.

Axiom A6 If each of x, y, and z is a number, then

$x * (y+z) = (x * y)+(x * z)$.

The Combining Axiom If x and y are numbers so that $x < y$, and w is a number, then $x+w < x+z$.

Chapter 8

Geometry, Ochoa

8.1 Introduction

The introduction is under development.

8.2 Theorem Sequence

We begin with some formal rules of logic. A statement is a sentence, which in a given context, is either true or false.

Logic Rule 1 *No unstated assumption may be used in a proof.*

Logic Rule 2 *Let p and q be statements. If the compound statement "p and not q" implies a contradiction, then the statement "If p, then q" is true.*

We may use Logic Rule 2 to prove the statement in the following example.

Example 3 *Let a be a positive integer. If 9 divides n, then 3 divides n.*

Proof. Suppose that 9 divides n and 3 does not divide n. Since 3 does not divide n, neither does 3^2. Since $3^2 = 9$, it follows that 9 cannot divide n. This is a contradiction, since 9 divides n. Therefore, if 9 divides n, then 3 divides n.

Logic Rule 4 *Let p be a statement. The negation of the statement "Not p" means the same as p.*

Logic Rule 5 *Lep p and q be statements. The negation of the statement "If p, the q" means the same as "p and not q.*

Logic Rule 6 *Let p and q be statements. The negation of the statement "p and q" means the same as "Not p or not q." The negation of the statement "p or q" means the same as "Not p and not q."*

Let $p(x)$ be a sentence about x (in this case x is calles a variable). We say that z is in the domain of p if the sentence $p(z)$ is a statement. For example, let $p(x)$ be the sentence "$x + 3 = 7$." Then 6 is in the domain of p since $p(6)$ is the (false) statement "$6 + 3 = 7$." On the under hand w is not in the domain of p, since we cannot, in the current context, determine whether "$w + 3 = 7$" is true or false.

Let $p(x)$ be a sentence about x. The sentence "For all x, $p(x)$" is a statement. The expression "for all" is called the universal quantifer. It is understood that the phrase "for all x" refers only to x in the domain of $p(x)$. The sentence "There exists an x such that $p(x)$" is a statement. The expression "there exists" is called the existensial quantifer. It is understood that the phrase "there exists an x" refers to some x in the domain of $p(x)$.

Logic Rule 7 *Let $p(x)$ be a sentence about x. The negation of the statement "For all x, $p(x)$" means the same as "There exists an x such that not $p(x)$."*

Logic Rule 8 *Let $p(x)$ be a sentence about x. The negation of the statement "There exists an x such that $p(x)$" means the same as "For all x, not $p(x)$."*

Logic Rule 9 *Let p and q be statements. Suppose the statements "If p, then q" and p are steps in a proof. Then q is a valid step.*

Logic Rule 10 *Let p, q, and r be statements. The following are all true statements:*

1. *If p, then p or q.*
   If q, then p or q.
2. *If p and q, then p.*
   If p and q, then q.
3. *If "Not q" implies "Not p," then p implies q.*
   If p implies q, then "Not q" implies "Not p".
4. *If p implies q and q implies r, then p implies r.*

Logic Rule 11 *For every statement p, either p or "Not p" is true.*

We next state some properties of equality.

Equality Properties 12 *Equality satisfies the following:*

Reflexive Property. $a = a$.

Symmetric Property. *If $a = b$, then $b = a$.*

Transitivity Property. *If $a = b$ and $b = c$, then $a = c$.*

The following terms will remain undefined: point, line, to lie on, between, congruent, plane, to pass through, and set.

Axiom 13 *For every pair of points P and Q, there exists a unique line passing through P and Q.*

Axiom 14 *For every line l, there exists at least two distinct point which lie on l.*

Axiom 15 *There exist three distinct points with the property that no line passes through all of them.*

Definition 16 *Two distinct lines are said to be parallel if no point lies on both lines.*

We are now ready for our first few propositions. The proofs are provided.

Proposition 17 *If l and m are distinct lines that are not parallel, then there is exactly one point lying on l and m.*

Proof. Suppose not. Let l and m be distinct lines that are not parallel such that there are two distinct points, P and Q, lying on both lines. By Axiom 13, P and Q determine a unique line. Thus, l and m must be the same line, a contradiction. □

Proposition 18 *For every line, there is a point not lying on it.*

Proof. Let l be a line. Let P, Q, and R be three distinct points with the property that no line passes through all three points. Axiom 15 garranties these point exist. At most, only two of these three points lie on l. It follows tht one of the points is not on l. □

Proposition 19 *For every point, there is at least one line passing through it.*

Proof. Let P be a point. Let Q be another point distinct from P (Axiom 15 guarantees the existance of Q). Let l be the line passing through P and Q (Axiom 13). Using Proposition 18, let R be a point not on l. Let m be the line passing through Q and R. Note that l and m are not parallel. By Proposition 17, Q must be the only point lying on both l and m. Thus, P does not lie on m. □

The proof of the next proposition is the first assignment.

Proposition 20 *There exist three distinct lines with the property that no point lies on all three lines.*

Let l and m be lines and let P be a point. If P lies on both lines, we say that l and m intersect (or meet) at P.

Let A, B, and C be points. We write $A * B * C$ to mean that B is between A and C.

Betweenness Axiom 21 *If $A * B * C$, then A, B, and C are three distinct points all lying on the same line and $C * B * A$.*

Let P and Q be distinct points. We write $\overleftrightarrow{PQ}$ to denote the line through P and Q. Note that $\overleftrightarrow{AB}$ and $\overleftrightarrow{BA}$ are the same line.

Betweenness Axiom 22 *Given any two points B and D, there exist points A, C, and E lying on $\overleftrightarrow{BD}$ such that $A * B * D$, $B * C * D$, and $B * D * E$.*

Betweenness Axiom 23 *If A, B, and C are distinct points lying on the same line, then exactly one of the points is between the other two.*

Let A and B be points. The line segment AB is defined to be the set of points A, B, and all points between A and B. The ray $\overrightarrow{AB}$ is the set of all points C such that $A * B * C$ together with all points on AB. Betweenness Axiom 22 guarantees that given points A and B, both AB and $\overrightarrow{AB}$ exist. Every point on AB is a point on $\overrightarrow{AB}$, and every point om $\overrightarrow{AB}$ is a point on $\overleftrightarrow{AB}$. That is, AB is a subset of $\overrightarrow{AB}$, and $\overrightarrow{AB}$ is a subset of $\overleftrightarrow{AB}$. Finally, note that AB and BA are the same set of points.

Lemma 24 *Let Q and B be points. Suppose that C is a point lying on both $\overrightarrow{AB}$ and $\overrightarrow{BA}$. Then, C lies on AB.*

Lemma 25 *Let A and B be points. Suppose that C is a point on $\overleftrightarrow{AB}$. Then, C lies on $\overrightarrow{AB}$ or on $\overrightarrow{BA}$.*

Hint. Either C is on $\overrightarrow{AB}$ or it isn't. If C is on $\overrightarrow{AB}$, We're done! Suppose C is not on $\overrightarrow{AB}$. What can you conclude?

Let U and V be sets of points. We say that U is a subset of V, and write $U \subseteq V$, if every point in U is also a point in V. We say that U and V are equal, and write $U = V$, if they are subsets of each other. The intersection of U and V, written $U \bigcap V$, is the set of all points which are in both U and V. The union of U and V, written $U \bigcup V$ is the set of all points which are in U or V. Note that $U \bigcap V \subseteq U$ and $U \subseteq U \bigcup V$.

Proposition 26 *Let A and B be points. Then*

1. $\overrightarrow{AB} \bigcap \overrightarrow{BA} = AB$, *and*
2. $\overrightarrow{AB} \bigcup \overrightarrow{BA} = \overleftrightarrow{AB}$.

Proof. To prove part 1, we must show that $\overrightarrow{AB} \bigcap \overrightarrow{BA} \subseteq AB$ and $AB \subseteq \overrightarrow{AB} \bigcap \overrightarrow{BA}$. We already know that $AB \subseteq \overrightarrow{AB}$ and that $AB \subseteq \overrightarrow{BA}$. Thus, $AB \subseteq \overrightarrow{AB} \bigcap \overrightarrow{BA}$. Lemma **??** guarantees that $\overrightarrow{AB} \bigcap \overrightarrow{BA} \subseteq AB$. Thus, $\overrightarrow{AB} \bigcap \overrightarrow{BA} = AB$.

To prove part 2, we must show that $\overrightarrow{AB} \bigcup \overrightarrow{BA} \subseteq \overleftrightarrow{AB}$ and $\overleftrightarrow{AB} \subseteq \overrightarrow{AB} \bigcup \overrightarrow{BA}$. We already know that $\overrightarrow{AB} \subseteq \overleftrightarrow{AB}$ abd $\overrightarrow{BA} \subseteq \overleftrightarrow{AB}$. Thus, $\overrightarrow{AB} \bigcup \overrightarrow{BA} \subseteq \overleftrightarrow{AB}$. Lemma **??** guarantees that $\overleftrightarrow{AB} \subseteq \overrightarrow{AB} \bigcup \overrightarrow{BA}$. Thus, $\overleftrightarrow{AB} = \overrightarrow{AB} \bigcup \overrightarrow{BA}$. □

Definition 27 *Let l be a line. Let A and B be two points that do not lie on l. We say that A and B are on the same side of l if the line segment AB does not intersect l (i.e. no point lies of both AB and l). If A and B are not on the same side of l, the we say A and B are on opposite sides of l.*

If A and B are not on l, then Logic Rule 11 guarantees that either A and B are on the same side of l or they are on opposite sides on l.

The next lemma guarantees that our geometry is two-demensional.

Betweenness Axiom 28 (Seperation) *For any line l and any three points A, B, and C not lying on l:*

1. *If A and B are on the smae side of l and B and C are on the same side of l, then A and C are on the same side of l*
2. *If A and B are on opposite sides of l and B abd C are on oposite sides of l, then A and C are on the same side of l.*

Let l be a line. Let P be a point not on l. The set of all points on the same side of l as P is called a half-plane determined by l.

Lemma 29 *Let l be a line and A a point not on l. Then, there exists a point B such that A and B are on opposite sides of l.*

Lemma 30 *Evrey line determines exactly two half-planes.*

Lemma 31 *Let l be a line. The two half-planes determined by l have no point in common.*

The following proposition follows immediatly from the two previous lemmas. No proof is necessary.

Proposition 32 *Every line determines exactly to half-planes and these half planes have no point in common.*

Proposition 33 *Let l be a line and let P be a point on l. Then there is a line, distinct from l, passing through P.*

Proposition 34 *Let l be a line. Suppose A and B are points on opposite sides of l. Let C be the point where AB and l intersect. Then $A * C * B$.*

Proposition 35 *Suppose $A * B * C$. Let l be a line distinct from $\overleftrightarrow{AC}$ passing through C. Then A and B are on the same side of l.*

Proposition 36 *Suppose that $A * B * C$ and $A * C * D$. Let l be a line, distinct from $\overleftrightarrow{AC}$, passing through C. The B and D are on opposite sides of l.*

Proposition 37 *Suppose that $A * B * C$ and $A * C * D$. Then $B * C * D$ and $A * B * D$.*

Proof. Let l be a line, distinct from $\overleftrightarrow{AC}$, passing through C. By Proposition 36, B and D are on opposite sides of l. By Proposition 34, $B * C * D$.

A similar argument shows that $A * B * D$. □

Proposition 38 *Let A, B, and C be points such that $C * A * B$ and let l be the line passing through A, B. and C. If P is a point on l, then P lies on $\overrightarrow{AB}$, or P lies on $\overrightarrow{AC}$.*

Proof. Let P be a point on l. Either P lies on $\overrightarrow{AB}$, or it does not. If P lies on $\overrightarrow{AB}$, then we are done! Suppose that P does not lie on $\overrightarrow{AB}$. Then, $P \neq A$, $P \neq B$, P is not between A and B, and B is not between A and P. Therefore, by Betweeness Axiom 23, $P * A * B$. Either $P = C$, or $P \neq C$. If $P = C$, we are done! Suppose that $P \neq C$. By Betweeness Axiom 23, either $A * P * C$, $A * C * P$, or $C * A * P$.

Suppose $C * A * P$ (We will show that this leads to a contradiction). By Betweeness Axiom 23, either $C * P * B$, $P * C * B$, or $P * B * C$. Suppose that $C * P * B$. then $B * A * P$ and $B * P * C$. Therefore, by Proposition 37, $A * P * C$. However, this contradicts the fact that $C * A * P$. Suppose that $P * C * B$. Then $B * A * C$ and $B * C * P$. Therefore, $A * C * P$. However, this also contradicts the fact that $C * A * P$. Finally, suppose that $P * B * C$. Then $P * A * B$ and $P * B * C$. Therefore, $A * B * C$. Again, this contradicts that fact that $C * A * B$. It follows that if we assume that $C * A * P$, that we get a contradiction. Thus $P * C * A$ or $C * P * A$. Therefore, P is on $\overrightarrow{AC}$. □

Definition 39 *Let A, B, and C be distinct points not lying on the same line. The set of points lying on AB, BC, or AC is called triangle ABC. We denote this triangle by $\triangle ABC$. The points A, B, and C are called vertices of $\triangle ABC$. the line segments AB, BC, and AC are called the sides of $\triangle ABC$. Note that $\triangle ABC$, $\triangle ACB$, $\triangle BAC$, $\triangle BCA$, $\triangle CAB$, and $\triangle CBA$ are the same set of points.*

Proposition 40 *A triangle exits.*

Theorem 41 (Pasch's Theorem) *Let A, B, and C be points. Let l be a line passing through AB at a point between A and B. Then l intersects AC or BC. Moreover, if C is not on l, then l intersects exactly one of AC or BC.*

Hint. Either l intersects BC or it does not. If l intersects BC, we are done! Suppose l does not intersect BC. Say something about the points B and C. Conclude something about the points A and C.

Proposition 42 *Let A, B, and C be points such that $A * B * C$. If P is a point on AC, then P lies on AB or on BC. (i.e. $AC = AB \bigcup BC$).*

Proposition 43 *Let A, B, and C be points such that $A * B * C$. Then B is the only point lying on both AB and BC. (i.e. $AB \bigcap BC = \{B\}$).*

Proposition 44 *Let A, B, and C be points such that $A * B * C$. Then B is the only point lying on both $\overrightarrow{BA}$ and $\overrightarrow{BC}$. (i.e. $\overrightarrow{BA} \bigcup \overrightarrow{BC} = \{B\}$).*

Proposition 45 *Let A, B, and C be points such that $A * B * C$. If P is a point on $\overrightarrow{AB}$, then P is also on $\overrightarrow{AC}$. Conversly, if P is on $\overrightarrow{AC}$, then P is also on $\overrightarrow{AB}$. (i.e. $\overrightarrow{AB} = \overrightarrow{AC}$).*

Definition 46 *Let A, B, and C be three distinct points not lying on the same line. We define angle CAB, and write $\angle CAB$, to be the set of points lying on $\overrightarrow{AB}$ or on $\overrightarrow{AC}$.*

Definition 47 *We say that the point is in the interior of the angle $\angle CAB$ if*

1. *D and B are on the same side of $\overleftrightarrow{AC}$, and*
2. *D and C are on the same side of $\overleftrightarrow{AB}$.*

Proposition 48 *Let $\angle CAB$ be an angle and let D be a point such that $B * D * C$. Then, D is a point in the interior of $\angle CAB$.*

Proposition 49 *Let $\angle CAB$ be an angle and let D be a point lying on $\overleftrightarrow{BC}$. If D is in the interior of $\angle CAB$, then $B * D * C$.*

Propositons 48 and 49 tell us that a point D on the line $\overleftrightarrow{BC}$ is in the interior of $\angle CAB$ if and only if $B * D * C$.

Here is something to ponder.

Conjecture 50 *Let D be a point in the interior of the angle $\angle CAB$. Is there a point E on $\overrightarrow{AB}$ and a point F on $\overrightarrow{AC}$ such that $E * D * F$?*

Proposition 51 *Let D be a point in the interior of $\angle CAB$. If E is a point on $\overrightarrow{AD}$, distinct from A, then E is in the interior of $\angle CAB$.*

Proposition 52 *Let D be a point in the interior of $\angle CAB$. Let E be a point on $\overleftrightarrow{AD}$ such that $E * A * D$. Then, no point on $\overrightarrow{AE}$ is in the interior of $\angle CAB$.*

Proposition 53 *Let D be a point in the interior of $\angle CAB$. Let E be a point on $\overleftrightarrow{AC}$ such that $E * A * C$. Then B is in the interior of $\angle DAE$.*

Definition 54 *Let A, B, and C be three distinct points not lying on the same line. Let D be a point distinct from A. We say that $\overrightarrow{AD}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AB}$ if D is in the interior of $\angle CAB$.*

Theorem 55 (Crossbar Theorem) *Suppose $\overrightarrow{AD}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AB}$ Then, $\overrightarrow{AD}$ intersects BC.*

Hint. Use Proposiiton 53.

Definition 56 *Let P be a point. We say that P is in the interior of each of the three angles $\angle CAB$, $\angle ABC$, and $\angle ACB$. If P is not in the interior of $\triangle ABC$ and does not lie on $\triangle ABC$, we say that P is in the exterior of $\triangle ABC$.*

Definition 57 *Let P be a point. We say that r is a ray emanating from P if there is a point Q such that r is the ray $\overrightarrow{PQ}$.*

Proposition 58 *Let P be a point in the exterior of $\triangle ABC$. Let r be a ray emamating from P that intersects AB at a point between A and B. Then r intersects AC or BC.*

Proposition 59 *Let P be a point in the interior of $\triangle ABC$ and let r be a ray emanating from P. Then r intersects at least one of the sides of the triangle; moreover, if r does not intersect a vertex, then r intersects exactly one side.*

We write $X \cong Y$ to mean X and Y are congruent.

Congruence Axiom 60 *Let A, B, and A' be any points such that A and B are distinct. Let r be any ray emanating from A'. Then there exists a unique point B' on r, distinct from A' such that $AB \cong A'B'$.*

Congruence Axiom 61 *If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Moreover, every segment is congruent to itself.*

Congruence Axiom 62 (Segment Addition) *If $A * B * C$, $A' * B' * C'$, $AB \cong A'B'$, and $BC \cong B'C'$, then $AC \cong A'C'$.*

Congruence Axiom 63 *Given $\angle BAC$ and distinct points A' and B', there exist points C' and D' on opposite sides of $\overleftrightarrow{A'B'}$ such that $\angle BAC \cong \angle B'A'C'$ and $\angle BAC \cong \angle B'A'D'$.*

Definition 64 *We say that $\triangle ABC$ and $\triangle A'B'C'$ are congruent, and write $\triangle ABS \cong \triangle A'B'C'$ if $\angle BAC \cong \angle B'A'C'$, $\angle ABC \cong \angle A'B'C'$, $\angle ACB \cong \angle A'C'B'$, $AB \cong A'B'$, $AC \cong A'B'$, and $BC \cong B'C'$.*

Congruence Axiom 65 (Side-Angle-Side) *Given $\triangle ABC$ and $\triangle A'B'C'$, if $AB \cong A'B'$, $AC \cong A'C'$, and $\angle BAC \cong \angle B'A'C'$, then $\triangle ABC \cong \triangle A'B'C'$.*

Congruence Axiom 66 *Let α, β, and γ be angles. If $\alpha \cong \beta$ and $\beta \cong \gamma$, then $\alpha \cong \gamma$. Moreover, every angle is congruent to itself.*

Proposition 67 *Let $\triangle ABC$ be a triangle such that $AB \cong AC$. Then $\angle ABC \cong \angle ACB$.*

Proposition 68 (Segment Subtraction) *If $A*B*C$, $D*E*F$, $AB \cong DE$, and $AC \cong DF$, then $BC \cong EF$.*

Hint. Using Congruence Axiom 60, let G be the unique point on $\overrightarrow{EF}$ such that $BC \cong EG$. Use Congruence Axiom 62 and Congruence Axiom 63 to show that $DG \cong DF$. Now use Congruence Axiom 60 again.

Proposition 69 *Suppose $AC \cong DF$. Let B be a point such that $A * B * C$. Then there is a unique point E between D and E such that $AB \cong DE$.*

Definition 70 *Let A, B, C, and D be points. We write $AB < CD$ if there is a point E, $C * E * D$, such that $AB \cong CE$.*

Proposition 71 *Let A, B, C, and D be points. Either $AB < CD$, $AB \cong CD$, or $AB > CD$.*

Proposition 72 *If $AB < CS$ and $CD \cong EF$, then $AB < EF$.*

Proposition 73 *If $AB < CD$ and $CD < EF$, then $AB < EF$.*

Definition 74 *Let α and β be angles. We say that α and beta are supplementary angles if there are points A, B, C, and D, such that $B * A * C$, D is not on $\overleftrightarrow{BC}$, $\alpha = \angle BAD$, and $\beta = \angle CAD$.*

Definition 75 *An angle is a right angle if it is congruent to one of its supplementary angles. That is, if α and β are supplementry angles and $\alpha \cong \beta$, then α is a right angle.*

Definition 76 *Let α and β be angles. We say that α and β are vertical angles if there exist points A, B, C, D, and E such that $B * A * C$, $D * A * E$, $\overleftrightarrow{BC}$ and $\overleftrightarrow{DE}$ are distinct lines, $\alpha = \angle BAE$, and $\beta = \angle CAD$.*

Chapter 9

Jordan-Curve Theorem, W. S. Mahavier

9.1 Introduction

This section was developed by Wm. S. Mahavier who adapted the material from the text, (forthcoming). It is a good example of how one can take a traditional book and break the material up into bite sized pieces for students to absorb. The sequence is probably best for students who have had at least an introduction to topology and is certainly appropriate at the graduate level.

9.2 Theorem Sequence

All point sets in this section are subsets of $\Re^2$. A rectangular grating is the union of a square with sides parallell to the axes and a finite collection of line segments each of which is either vertical or horizontal and has both end points on the square. The 2-cells of a rectangular grating $\mathcal{G}$ are the closures of the components of $\Re^2 - \mathcal{G}$ (one of which is not really a 2-cell.) The 1-cells of $\mathcal{G}$ are the sides of the bounded 2-cells of $\mathcal{G}$. The 0-clells of $\mathcal{G}$ are the corners of the boudned 2-cells of $\mathcal{G}$.

Theorem 1 *If H and K are disjoint closed subsets of $\Re^2$ and H is bounded, then there is a grating $\mathcal{G}$ such that no 2-cell of $\mathcal{G}$ intersects both H and K.*

A k-chain on a grating $\mathcal{G}$ is a function from the set of k-cells of $\mathcal{G}$ into the set $\{0, 1\}$. Obviously this is equivalent to choosing a subcollection of the k-cells of $\mathcal{G}$. If each of C and D is a k-chain then C+D is the k-chain such that (C+D)(M) = 0 if and only if C(M) = D(M) = 1 or C(M) = D(M) =0.

The k-chaing of a grating $\mathcal{G}$ with this operation are denoted C_k(G). The k-chain which is 1 only at the k-cell K will be denoted $\tilde{K}$.

Theorem 2 *The k-chains on a grating $\mathcal{G}$ form a commutative group.*

Theorem 3 *There is a homomorphism δ_2 from $C_2(G)$ into $C_1(G)$ such that if K is a 2- cell of the grating $\mathcal{G}$ then $\delta_2(\tilde{K})$ is 1 only at the 1-cells which are subsets of K. Moreover, if C is a 2-chain and L is a 1-cell, then $\delta_2(C)(L) = 1$ if and only if there is an odd number of 2-cells K such that C(K) = 1 and K contains L.*

Theorem 4 *There is a homomorphism δ_1 from $C_1(G)$ into $C_0(G)$ such that if K is a 1-cell of the grating $\mathcal{G}$ then $\delta_1(\tilde{(K)})$ is 1 only at the 0-cells which are subsets of K. Moreover, if C is a 1-chain and L is a 0-cell, then $\delta_1(C)(L) = 1$ if and only if there is an odd number of 1-cells K such that C(K) =1 and K contains L.*

We will ignore the subscript and denote both δ_2 and δ_1 by δ. These homomorphisms are called boudnary operators and for the k-chain C, δC is called the boundary of C.

The statement that a k-chain C is a cylce means that $k = 0$ or $k = 1$ and $\delta C = 0$ or $k = 2$ and $\delta C = 0$.

Theorem 5 *The k-cycles form a subgroup of $C_k(G)$.*

Theorem 6 *If k is positive and C is a k-chain then δC is a (k-1) -cycle.*

A cycle is said to be a bounding cycle if it is in the boundary of a chain.

Theorem 7 *If C is a bounding 0-cycle, there are an even number of 0-cells K such that C(K) = 1.*

Theorem 8 *If D is a 1-cycle and $|D|$ is finite then $\ldots \ni$???*

If C is a k-chain, the carrier of C, denoted $|C|$, is the union of all k-cells K such that C(K)=1. A k-chain C is said to be connected if $|C|$ is connected. A k-chain D is said to be a component of a k-chain C if D is connected and $|D|$ is a component of $|C|$. A k-chain C is said to in a pont set M is $|C|$ is a subset of M.

Theorem 9 *If D is a component of a k-chain C and k is positive then $|\delta D|$ is $|\delta C| \cap |D|$.*

Theorem 10 *If C is a 1-chain and $|\delta C|$ is a set consisting of two points p and q, then p and q are in the same component of $|C|$.*

Theorem 11 *If C is a 2-chain, $|\delta C|$ is the point set boundary of $|C|$.*

If each of $\mathcal{G}$ and H is a grating then H is said to be a refinement of $\mathcal{G}$ is H contains $\mathcal{G}$.

Theorem 12 *If $\mathcal{G}$ and H are gratings, H is a refinement of $\mathcal{G}$ and C is a k-china on $\mathcal{G}$, then there is only one k-chain D on H such that $|C| = |D|$.*

Definition 13 *Suppose each of $\mathcal{G}$ and H is a grating an H is a refinement of $\mathcal{G}$. For each k-chain C on $\mathcal{G}$, sdC denote the c-chian on H such that $|sdC| = |C|$. The chain sdC is called a subdivision of C.*

Theorem 14 ***Each 1-cycle is the boundary of exactly two 2-chains.*

Suppose M is a point set and C is a k-cycle on a grating $\mathcal{G}$. Then C bounds in M means that there is a refinment H of $\mathcal{G}$, and a (k+1)-chain D on H such that $|D|$ is a subset of M and $\delta D = sdC$.

Theorem 15 *Suppose each of $C_1, C_2, \ldots, C_2$ is a k-cycle on the grating $\mathcal{G}$ which bounds in the points sets M. Then $C_1 + C_2 + \ldots + C_n$ bounds in M.*

Theorem 16 *If the k-cycle C does not bound in the points set M then some component of C does not bound in M.*

Theorem 17 *If $\Re^2 - M$ is connected then every 1-cycle in M bounds in M.*

Lemma 18 *Suppose U is an open set and p is a point of U. The set of all points q of U such that there is a grating $\mathcal{G}$ and a 1chain on $\mathcal{G}$ in U whose boundary is $\tilde{p} + \tilde{q}$ is both open and closed in U.*

Theorem 19 *Supose U is an open set, p and q are two points of U, p and q are 0-cells of the grating $\mathcal{G}$, and C is $\tilde{p} + \tilde{q}$. Then C bounds in U if and only if p and q are in the same omponent of U.*

Theorem 20 *If C and D are 1-cycles and p and q are two points no in $|C| \cup |D|$, then at least one of the cycles C, D, and C+D bounds in $\Re^2 - \{p, q\}$.*

Two 1-chains C and D on a grating $\mathcal{G}$ are said to have general intersection if
(1) no 0-cell common to $|C|$ and $|D|$ is the outer edge of $\mathcal{G}$ and
(2) at each 0-cell p common to $|C|$ and $|D|$ the horizontal 1-cells of $\mathcal{G}$ containing p are contained in only one of $|C|$ and $|D|$ and the vertical 1-cells of $\mathcal{G}$ containing p are contained in only one of $|C|$ and $|D|$.

Theorem 21 *Suppose C is a 1-cycle and D is a 1-chain having general intersection with C and whose boundary is $\tilde{p} + \tilde{q}$. Then C bounds in $\Re^2 - \{p, q\}$ if and only if $|C|$ and $|D|$ intesect at an even number of 0-celss.*

Theorem 22 *If two 1-cycles C and D have general intesection then $|C|$ and $|D|$ intersect at an even number of 0-cells.*

Theorem 23 *Suppose C and D are 1-chains having general intersection, $\delta C = \tilde{p} + \tilde{q}$, and $\delta D = \tilde{r} + \tilde{s}$, $|C| \cap |D|$ contains an odd number of 0-cells and M is a continuum which contains p and q but does not intersect $|D|$. Then $M \cup |C|$ separates r and s.*

Lemma 24 *Suppose U and V are open sets, U is bounded, $\mathcal{G}$ is a grating and C is a chain on $\mathcal{G}$ such that $|C| \subseteq U \cup V$. Then there is a refinement H of $\mathcal{G}$ such that every cell of H which is contained in $|sdC|$ is either a subset of U or a subset of V.*

Theorem 25 *Suppose U and V are upen sets in* $\Re^2$*, U is bounded, C and D are 1-chains such that C is in U, D is in V, and* $\delta C = \delta D = \tilde{p} + \tilde{q}$*, and C+D bounds in* $U \cup V$*. Then* $\tilde{p} + \tilde{q}$ *bounds in* $U \cap V$*.*

Theorem 26 *Suppose M and N are closed subsets of* $\Re^2$ *and M is bounded or M and N are disjoint. If* $\delta C = \delta D = \tilde{p} + \tilde{q}$*,* $|C|$ *does not intersect M, and* $|D|$ *does not intersect N but C+D bounds in* $\Re^2 - (M \cap N)$*, then p and q are not separated by* $M \cup N$*.*

Theorem 27 *Suppose that M and N are closed subsets of* $\Re^2$ *and either M and N are disjoint or M is bounded and* $M \cap N$ *is connected. If p and q are not separated by eiether M or N then P and q are not separated by* $M \cup N$*.*

Theorem 28 *If U and V are connected open sets whose union is* $\Re^2$ *then the intersection of U and V is connected.*

Theorem 29 *Suppose M is a closed set contained in a connected open set U in* $\Re^2$ *and* $V_1, V_2, \ldots$ *are components of the point set* $\Re^2 - M$*. Then the components of U-M are* $U \cap V_1, U \cap V_2, \ldots$*.*

Theorem 30 *Suppose U and V are open subsets of* $\Re^2$ *which do not separate* $\Re^2$*, K and L are 1-chains such that* $|K|$ *is a subset of U and* $|L|$ *is a subset of V and* $\delta K = \delta L$*. If* δK *bounds in* $U \cap V$ *then K+L bounds in* $U \cup V$*.*

Theorem 31 *Supopse M and N are connected closed subset of* $\Re^2$ *and C is a 0-cycle which bounds a 1-chain K such that* $|K|$ *does not intersect M and bounds a 1-chain L such that* $|L|$ *does not intersect N. Suppose the 2n 0-cells of* $|C|$ *are listed in some way as* $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$*. If K+L does not bouind in* $\Re^2 - (M \cap N)$*, the for some r,* $M \cup N$ *separates* x_r*from* y_r*.*

Theorem 32 *No arc separates* $\Re^2$*.*

Theorem 33 *If J is a simple closed curve in* $\Re^2$ *then* $\Re^2 - J$ *is the union of two open sets each of which has J as its point set boundary.*

Theorem 34 *If each of M and N is a closed connected subset of* $\Re^2$ *and M is bounded, but* $M \cap N$ *is not connected, there is a pair of points separated by* $M \cup N$ *but not by M.*

Theorem 35 *The unit 2-cell in* $\Re^2$ *is unicoherent, that is, if M and N are closed connected sets whose union is* $\Re^2$ *then* $M \cap N$ *is connected.*

Chapter 10

Place-Value Model for the Numbers, Parker

10.1 Introduction to the instructor

These notes are intended for a course in which students may be proving theorems on their own for the first time. The mathematical content addresses the question "Can we give meanings for *number*, $<$, $+$, and $*$ so that the axioms for the numbers are consequences of the definitions?"

In these notes, a place-value model is pursued. The students are given that the natural numbers exist and functions from the natural numbers into a two-element set become the objects of study.

The problem set is designed so that even a class that plods will be exposed to ideas of comparing sets and imposing an order on a set. A class that experiences success from the beginning can be expected to get at least to the point of recognizing that the model is Dedekind complete and experience the difficulties of imposing an algebra driven by algorithms that are not universally defined. I teach these notes as a one-semester course. I routinely have classes that prove the order complete, and get some of the set comparisons, but I have never had a class get entirely through the imposition of the algebra on the model. These notes stop short of an entire model for the numbers. Students that can get through most of these problems will have the maturity to do mathematics of more importance (perhaps computer science majors could derive major benefit) than what would follow. Nevertheless, for a teacher determined to pursue this inquiry to its "completion," I have given a brief description of what might come next at the end of Section 10.5.

The story line for the course goes like this:

- Defining a place-value number consists of specifying the digit in each place value.
- With the comparison principle "find the first place-value in which the objects are different and use it for comparison" an order is defined, but it contains pairs of elements so that there is no object between them.
- Having corrected this flaw, the set admits an order with no minimum and no maximum and with the Dedekind cut property.
- Meanwhile, we are also counting sets. The natural numbers are shown to be infinite.
- The natural numbers are shown to be as large as the set of elements whose "partners" were purged to eliminate the "holes" and this set is shown to be dense in the order.
- Segments are shown to be as large as the entire set.
- The natural numbers are shown to be not as large as this set or, for that matter, any Dedekind-complete set.

- Using the addition algorithm from grade school arithmetic, and being careful, an algebra is imposed.
- With this algebra, the set admits an order-preserving local semigroup which lacks an identity, but in which one can solve all of the "tractable" equations.

Since progress through these notes depends on the students finding the conjectures which are not theorems and then addressing the issues raised in trying to produce structures about which the conclusions are true, the instructor must exercise some care in when access to subsequent problems is granted to the students. Also, students often uncover theorems while working on problems and theorems can often be sifted out of students' arguments; these make nice addenda to the notes. I have included some examples from a past class of mine in the Section 10.4. Section 10.5 contains remarks about particular problems or definitions and possible timing schemes for presenting the problems. The order in which I have listed the problems need not be the optimal sequence for a particular class.

When I first taught a course of this type, I began by giving some background in logic. I no longer do this. The approach I use now is to give out a sheet on quantification when the course begins and deal with points of logic as they arise. When students argue correctly, they give lectures as good as yours. Particular points of logic may be emphasized by, after students finish arguments, going back and focusing on a part of the argument that used logic in a particular way. Correcting logically impaired arguments affords great opportunities for teaching the logic; the mistakes that students make often reflect the misunderstandings of other students in the class.

The instructor will also have available the opportunities that occur as students deal with the set of natural numbers, which is assumed to exist along with its arithmetic, and in terms of which the objects of the model are defined. In this course, the students typically address structures of the natural numbers as they work on the counting problems. The notions of odd and even, the fact that $\mathbf{N}$ is well-ordered by $<$, the infinitude of the primes, and the "uniqueness" of a prime factorization of a natural number routinely appear in the students' work. I usually demand a definition for odd and even, since the experience of formalizing "can be written as ..." affords an opportunity for the student to consciously use quantification. Discussion of $<$ naturally occurs the first time finite induction is used (or is appropriate). I let the class have unique prime factorization; a student clever enough to construct a counting argument based on unique factorization deserves to be rewarded. Once the class has shown that a proper subset may command its superset, I am willing to grant the infinitude of the primes, and even to share an argument for it.

I give only one test in this course, the final examination. I offer one credit each time a student presents an argument for a problem that the class judges as being correct. If a student has a problem that someone else presents, that student is allowed to turn in her/his write-up at the end of the class period in which the problem was finished. If the write-up is correct, the student gets one-half of a credit. The final examination is given as a take-home, "use your notes but nobody else's," test. It consists of a section of problems solved during the semester and a section of problems that the students have not yet solved and may not even have seen. Successful proofs for the problems proven during the course allow students to earn a grade of "C" or to keep whatever grade their work during the semester warranted. Successful work on the second section allows a student to increase her/his grade or to atone for slip-ups on the first section.

10.2 Introduction to students - An Introduction to Doing Mathematics

In this course, not only will you be responsible for understanding why the mathematics we cover is correct, but the responsibility for discovery will also be assigned to the class. One of the immediate results of this responsibility for doing mathematics yourself rather than just learning how someone else did it will likely be an acute awareness of the difference between the challenge associated with understanding why something is correct and discovering for yourself whether or not a conjecture is a theorem.

Doing mathematics can be extremely exhilarating when one succeeds in the discovery process; failing to do mathematics when one is putting in the time trying to do mathematics can be extremely frustrating. This introduction is designed to alert you to some tips that are designed to optimize the chances for success.

First, you must put in the time necessary to give your creative intelligence a chance to work. Flashes of insight typically occur after information is organized and mulled over. Commitment to solving problems often leads to help from the subconscious. Students often tell me that they got "the big idea" while walking across campus or after turning in for the night.

Second, solutions to problems need not come all at once. You may need to solve many small problems on the way to proving a theorem or disproving an incorrect conjecture. Some of the most important work in mathematics is the creation of technique. Take pride in progress toward a goal as well as reaching the goal. Any information you uncover is more than you knew before, and solving a problem is usually just a matter of putting together enough small solutions to allow you to see why the big problem is correct.

Students often tell me that they would be glad to put in the time if they just knew where to start. The following scheme is offered toward that end.

The awareness stage

1. Identify all the words in the problem and make sure that you *know* the definition of each of them. Try to recall examples that have dealt with these notions before. If a definition is new, make some examples for the definition.

2. Identify any theorems that may have already dealt with ideas present in the problem. Put techniques that gave rise to proofs in those contexts firmly in mind.

The direct approach

3. Make an example that models the hypothesis to the problem and try to show that the example exhibits the properties of the conclusion. (If you can prove that your example fails to have the properties of the conclusion, you will have shown that the problem is not a theorem!)

4. See if what allowed you to establish the conclusion in the example is a property of all examples covered by the hypothesis. If it is, write a proof. If not,

5. ...make an example which models the hypothesis but fails to have whatever special properties you used to get the conclusion in the previous example. Go to 3.

The indirect, or contrapositive approach

6. Suppose that the conclusion is false and try to show that the hypothesis must be false as well. If the problem is not a theorem, any conclusions you get must be qualities an example that disproves the conjecture must have.

7. Try to be aware of properties that, if they were added to the hypothesis, would guarantee the conclusion. Alternatively, you might also try to find conclusions that follow from the hypothesis, even if they do not include the one you seek. Even if you are not able to solve the problem as stated, you may be able to create a substitute theorem.

The main mindset is to be aware that even when arguments do not come quickly or easily, the hunt itself may be an important learning experience. Working on problems yourself is the central ingredient. Not only will it provide you with theorems that are "your own," but even when someone beats you to a solution, it will put you in a much stronger position to analyze the argument given.

A theory of sets and ordered pairs

We will not create an axiomatic set theory. Following, however, is an idiomatic presentation of some conventions that axiomatic set theory implies. We presuppose the existence of formal English as a language for expressing properties.

The primitive words are set, element, ordered pair, first co-ordinate, and second co-ordinate.

i. A set consists of an element or elements.

ii. An element of a set and the set consisting of that element are different objects.

iii. A set is defined by stating the properties its elements have. (The plural has been chosen here, but the definition of a set may be made by stating a single property and a set may have a single element.)

iv. Given a definition for a set, any object having the properties specified is an element of the set; and any element of the set has the properties specified in the definition.

v. An ordered pair consists of a first co-ordinate and a second co-ordinate.

vi. The first co-ordinate of an ordered pair may be the same set-theoretic object as the second co-ordinate, but as a part of the ordered pair, being the first co-ordinate is distinguishable from being the second co-ordinate.

We reserve a notation for the creation of definitions of sets and for defining ordered pairs.

Reserved symbols for definitions of sets are { : }. A symbol is created to follow the open brace and precede the colon and then properties that an element must have are stated in terms of that symbol after the colon and before the closed brace. Thus

$$\{x : x \text{ is a number and } x > 5\}$$

stands for "the set to which an element belongs provided that it is a number and it is greater than 5."

Reserved symbols for definitions of ordered pairs are (,). The first co-ordinate of the ordered pair is written after the open parenthesis and before the comma; the second co-ordinate of the ordered pair is written after the comma and before the closed parenthesis. Thus $(p, 5)$ stands for the ordered pair whose first co-ordinate is p and whose second co-ordinate is 5.

The purpose of this course is to build a model for the numbers. Our ultimate goal is to prove that the statements which are typically taken as axioms for the numbers are theorems in our model. In an axiomatic treatment, *number*, $<$, $+$, and $*$ are taken as primitive words; thus we provide definitions within the model so that if they are interpreted as the primitive words, the statements made by replacing the primitive words in the axioms with their analogues in the model become the topics of consideration.

You may assume that the natural numbers exist and have whatever properties number theory says they do. If there is doubt about a property of the natural numbers, we will either prove the property or indicate what property we are assuming.

10.3 Problem Sequence

Definition 1 Suppose that each of X and Y is a set. The statement that f is a **function from X into Y** means that f is a set so that

i. each element of f is an ordered pair whose first co-ordinate is an element of X and whose second co-ordinate is an element of Y; and

ii. if p is an element of X, then there is an element of f whose first co-ordinate is p; and

iii. if p and q are elements of f, then the first co-ordinate of p is not the first co-ordinate of q.

Notation: If f is a function from X into Y and (p,q) is an element of f, then we may write f(p) = q.

Definition 2 Suppose that each of X and Y is a set and that f is a function from X into Y. The statement that **M is the range of f** means that M is the set to which an element belongs provided that there is an element of f of which it is the second co-ordinate.

Problem 1 Suppose that X is a set with more than one element[1]. Show that the set to which an element belongs provided that it is an ordered pair whose first co-ordinate is an element of X and whose second co-ordinate is an element of X is not a function from X into X.

Definition 3 Suppose that X is a set and that L is a set each element of which is an ordered pair whose first co-ordinate is an element of X and whose second co-ordinate is an element of X. The statement that **L is an order on X** means that

i. if p is an element of X, then (p,p) is not an element of L; and

ii. if p and q are elements of X, then (p,q) is an element of L or (q,p) is an element of L; and

iii. if (p,q) and (q,r) are elements of L, then (p,r) is an element of L.

[1]That X has more than one element means that if p is an element of X, then there is an element of X different from p.

Problem 2 Suppose that X is a set with exactly one element. Show that there is no order on X.

Problem 3 Suppose that X is a set with more than one element and that L is an order on X. Show that L is not a function from X into X.

Definition 4 U is the set to which an element belongs provided that it is a function from the natural numbers into {0,1} so that its range is {0,1}.

Definition 5 Suppose that f and g are elements of U. The statement that **f precedes g** means that if n is the smallest natural number in $\{k : f(k) \neq g(k)\}$, then f(n) = 0 and g(n) = 1.

Definition 6 G = {(p,q) : p is an element of U, and q is an element of U, and p precedes q}.

Problem 4 Suppose that x is an element of U. Show that there is an element of U, call such an element y, so that (x,y) is an element of G.

Problem 5 Suppose that x is an element of U. Show that there is an element of U, call such an element y, so that (y,x) is an element of G.

Problem 6 Suppose that x and y are elements of U and (x,y) is an element of G. Show that there is an element of U, call such an element w, so that (x,w) and (w,y) are elements of G.

Problem 7 Show that G is an order on U.

Definition 7 Suppose that each of X and Y is a set. The statement that **X commands Y** means that there is a function from X into Y whose range is Y.

Problem 8 Show that U commands the natural numbers.

Definition 8 Suppose that each of X and Y is a set. The statement that **X is a subset of Y** means that if p is an element of X, then p is an element of Y.

Problem 9 Suppose that each of X and Y is a set and that X is a subset of Y. Show that Y commands X.

Problem 10 Suppose that X and Y are sets and that X is a subset of Y. Show that it is not the case that X commands Y.

Definition 9 Suppose that each of X and Y is a set and that there is an element of X which is an element of Y. The **intersection of X with Y** is {x : x is an element of X and x is an element of Y}.

Notation: X ∩ Y stands for "the intersection of X with Y."

Definition 10 Suppose that X is a set, L is an order on X, and a and b are elements of X so that (a,b) is an element of L, and there is an element of X, call such an element c, so that (a,c) is an element of L and (c,b) is an element of L. The **segment from a to b by L** is {x :(a,x) is an element of L and (x,b) is an element of L}.

Notation: If (a,b) is an element of the order L, (a,b) stands for "the segment from a to b by L."

Problem 11 Suppose that x is an element of U. Show that there is a segment by G so that x is an element of it.

Problem 12 Suppose that X is a set, L is an order on X, (p,q) and (r,s) are segments by L, and x is an element of (p,q) ∩ (r,s). Show that (p,q) ∩ (r,s) is a segment by L.

Problem 13 Suppose that x and y are elements of U and that G is an order on U. Show that there are segments by G, call them P and Q, so that

i. x is an element of P,

ii. y is an element of Q, and

iii. if w is an element of P, then w is not an element of Q.

Definition 11 Suppose that each of X and Y is a set. The **union of X with Y** is {p : p is an element of X or p is an element of Y}.

Notation: X ∪ Y stands for "the union of X with Y."

Definition 12 Suppose that X is a set, L is an order on X, and T and V are subsets of X. The statement that **(T,V) is a cut of X by L** means that

i. $T \cup V = X$; and

ii. if x is an element of T and y is an element of V, then (x,y) is an element of L.

Problem 14 Suppose that X is a set, L is an order on X, and (A,B) is a cut of X by L. Show that if p is an element of A, then p is not an element of B.

Problem 15 Suppose that X is a set and L is an order on X. Show that there is a cut of X by L.

Problem 16 Suppose that X is a set, L is an order on X, and (p,q) is an element of L. Show that there is a cut of X by L, call it (A,B), so that p is an element of A and q is an element of B.

Definitions 13 Suppose that X is a set, L is an order on X, p is an element of X, and M is a subset of X. The statement that **p is the max of M by L** means that p is an element of M, and if q is an element of M different than p, then (q,p) is an element of L. The statement that **p is the min of M by L** means that p is an element of M, and if q is an element of M different than p, then (p,q) is an element of L.

Definition 14 Suppose that X is a set and L is an order on X. The statement that **L has the Dedekind cut property** means that if (A,B) is a cut of X by L, then

i. A has a max by L or B has a min by L and

ii. it is not the case that both A has a max by L and B has a min by L.

Problem 17 Suppose that $L = \{(x,y) :$ x is a natural number, y is a natural number, and $x < y\}$. Show that L does not have the Dedekind cut property.

Definition 4' U' = {x : x is an element of U; and if n is a natural number, then there is a natural number greater than n, call it m, so that $x(m) = 1\}$.

Definition 6' G' = {(p,q) : p is an element of U', q is an element of U', and p precedes q}.

Problem 4' Suppose that x is an element of U'. Show that there is an element of U', call such an element y, so that (x,y) is an element of G'.

Problem 5' Suppose that x is an element of U'. Show that there is an element of U', call such an element y, so that (y,x) is an element of G'.

Problem 6' Suppose that x and y are elements of U' and (x,y) is an element of G'. Show that there is an element of U', call such an element w, so that (x,w) and (w,y) are elements of G'.

Problem 7' Show that G' is an order on U'.

Problem 8' Show that U' commands the natural numbers.

Problem 11' Suppose that G' is an order on U' and that x is an element of U'. Show that there is a segment by G' so that x is an element of it.

Definition 15 D $=$ {x : x is an element of U' and there is a natural number, call such a natural number n, so that if k is a natural number greater than n, then x(k) $=$ 1}

Problem 18 Suppose that x and y are elements of U' and (x,y) is an element of G'. Show that there is an element of D, call such an element w, so that (x,w) and (w,y) are elements of G'.

Problem 19 Show that the natural numbers commands D.

Problem 20 Show that U' commands U.

Problem 21 Show that G' has the Dedekind cut property.

Problem 22 Suppose that C is a function from the natural numbers into U' and that x $=$ {(p,q) : p is a natural number, q is an element of {0,1}, and q is not C(p)(p)}. Show that x is not an element of the range of C.

Problem 23 Show that the natural numbers do not command U'.

Problem 24 Suppose that x is an element of U' and n is a natural number. Show that {(p,q) : p is a natural number; and if $p < n$, then q $=$ x(p), or if $p = n$, then $q = 0$, or if $p > n$, then $q = 1$} is an element of U'.

Problem 25 Suppose that (x,y) is an element of G'. Show that (x,y) commands U'.

Definition 16 Suppose that X is a set, L is an order on X, and (p,q) is an element of L. The **interval from p to q by L** is {x : x is an element of (p,q), or x is p, or x is q}.

Notation: If (p,q) is an element of the order L, [p,q] stands for "the interval from p to q by L."

Problem 26 Suppose that M = {x : there is an element of G', call it (p,q), so that x = [p,q]}. Show that there is a function from the natural numbers into M, call such a function f, so that if n is a natural number then, then f(n+1) is a subset of f(n).

Problem 27 Suppose that (A, B) is an element of G' and x is an element of U' so that x is an element of (A,B). Show that there is an element of G', call it (p,q) so that [p,q] is a subset of (A,B) and x is not an element of [p,q].

Problem 28 Suppose that M = {x : there is an element of G', call it (p,q), so that x = [p,q]}, s is a function from the natural numbers into M so that if k is a natural number, then s(k+1) is a subset of s(k), and A = {x : there is a natural number, call it k, so that if p is an element of s(k), then (x,p) is an element of G'}. Show that (A,{x : x is an element of U' and x is not an element of A}) is a cut of U' by G'.

Problem 29 Suppose that M = {x : there is an element of G', call it (p,q), so that x = [p,q]}, and s is a function from the natural numbers into M so that if k is a natural number, then s(k+1) is a subset of s(k). Show that there is an element of U', call it w, so that if k is a natural number, then w is an element of s(k).

Problem 30 Suppose that X is a set and that L is an order on X so that L has the Dedekind cut property. Show that the natural numbers do not command X.

Definition 17 pc = {((0,0),0),(0,0)),((0,1),0),(0,1)),((1,0),0),(0,1)),((1,1),0),(1,0)), ((0,0),1),(0,1)),((0,1),1),(1,0)),((1,0),1),(1,0)),((1,1),1),(1,1))}

Problem 31 Show that pc is a function from

$\{$(x,y) : x is an element of $\{$(p,q) : p is an element of $\{0,1\}$ and q is an element of $\{0,1\}\}$ and y is an element of $\{0,1\}\}$ into $\{$(x,y) : x is an element of $\{0,1\}$ and y is an element of $\{0,1\}\}$.

Definitions 18 Suppose that (x,y) is an ordered pair. The **projection of (x,y) into its first co-ordinate** is x, and the **projection of (x,y) into its second co-ordinate** is y.

Notation: Suppose that p is an ordered pair. Π_1p stands for "the projection of p into its first co-ordinate," and Π_2p stands for "the projection of p into its second co-ordinate."

Definition 19 Suppose that m is a natural number and each of x and y is a function from $\{$k : k is a natural number and k $\leq$ m$\}$ into $\{0,1\}$.
ρ((x,y))(m) = $\Pi_2$pc((x(m),y(m)),0)), and
if t is a natural number so that t $<$ m, and
ρ((x,y))(t+1) = $\Pi_2$pc((x(t+1),y(t+1)),w)), then
ρ((x,y))(t) = $\Pi_2$pc(((x(t),y(t)), $\Pi_1$pc(((x(t+1),y(t+1)),w)).

Problem 31 Suppose that m is a natural number and each of x and y is a function from $\{$k : k is a natural number and k $\leq$ m$\}$ into $\{0,1\}$. Show that ρ((x,y)) is a function from $\{$k : k is a natural number and k $\leq$ m$\}$ into $\{0,1\}$.

Definition 20 & = $\{$((x,y),z) : each of x and y is an element of U', and z is an element of U' so that if m is a natural number, then there is a natural number, call it n, so that if n' $>$ n, then $\{$(s,t) : s $\leq$ m and t = z(s)$\}$ $\subset$
ρ(($\{$(s,t) : s $\leq$ n' and t = x(s)$\}$,$\{$(s,t) : s $\leq$ n' and t = y(s)$\}$)), and if
ρ(($\{$(s,t) : s $\leq$ n' and t = x(s)$\}$,$\{$(s,t) : s $\leq$ n' and t = y(s)$\}$))(1) = pc(((x(1),y(1)),w)),
then $\Pi_1$pc((x(1),y(1)),w)) = 0$\}$.

Problem 32 Show that & is a function from $\{$(x,y) : x is an element of U' and y is an element of U'$\}$ into U'.

Problem 33 Suppose that ((x,y),z) is an element of &. Show that ((y,x),z) is an element of &.

Problem 34 Suppose that & is a function from $\{$(x,y) : x is an element of U' and y is an element of U'$\}$ into U' and that each of x, y, and z is an element of U'. Show that &((x,&((y,z)))) = &((&((x,y)),z)).

Problem 35 Show that there is an element of U', call it z, so that if x is an element of U', then ((x,z),x) is an element of &.

Problem 36 Suppose that x and y are elements of U' so that ((x,y),z) is an element of &. Show that (x,z) is an element of G'.

Problem 37 Suppose that x and y are elements of U' so that (x,y) is an element of G'. Show that there is exactly one element of U', call it w, so that ((x,w),y) is an element of &.

Definition 21 E = {(x,y) : there is an element of U', call it z, so that ((x,y),z) is an element of &}

Problem 32' Show that & is a function from E into U'.

Problem 34' Suppose that (x,&((y,z)) is an element of E and (&((x,y)),z) is an element of E. Show that &((x,&((y,z)))) = &((&((x,y)),z)).

Problem 38 Suppose that (x,y) is an element of E and (w,x) is an element of G'. Show that (w,y) is an element of E.

10.4 Examples

Following are some theorems that students proved during a past offering of the course for which a selection from the problems above formed the corpus from which they worked. The problem from the notes which each addresses is noted in parentheses.

Theorem (6') Suppose that x and y are elements of U', x precedes y, n is the least natural number so that x(n) $\neq$ y(n), and p is a natural number greater than n so that y(p) = 1. Then {(s,t) : s is a natural number; and if s $\neq$ p, then t = y(s), or if s = n, then t = 0} is an element of U' so that x precedes it and it precedes y.

Theorem (21) Suppose that (P,Q) is a cut of U' by G'. Then if P has a maximum by G', then Q does not have a minimum by G; or if Q has a minimum by G', then P does not have a maximum by G'

Theorem (21) Suppose that (P,Q) is a cut of U' by G', n is a natural number, and s is a function from {k : k is a natural number no greater than n} into {0,1} so that

i. s(n)=0:

ii. if a is an element of P; then
{(j,k) : j is a natural number no greater than n and k = a(j)} = s, or there is an element of P, call such an element a', so that
{(j,k) : j is a natural number no greater than n and k = a'(j)} = s and a precedes a': and

iii. if b is an element of Q; then b(n) = 1, or there is an element of Q, call such an element c, so that c(n) = 1 and c precedes b.

Then
$\{(j,k) : j$ is a natural number; and if $j \leq n,$ then $k = s(j),$ or if $j > n, k = 1\}$ is an element of U' so that if a is an element of P different than it, then a precedes it, and if b is an element of Q different than it, then it precedes b.

Theorem (25) (The class called this the "bead-chain" theorem.) Suppose that x and y are elements of U', x precedes y, a is the least natural number so that x(a) is not y(a), b is a natural number so that b > a and y(b) = 1, and
ES = {p : there is an element of U', call it w, so that
p = {(c,d) : c is a natural number, and
if $c < b$, then d = y(c), or
if $c = b$, then d = 0, or
if there is a natural number,
call it e, so that c = b+e, then d = w(e)}}.
Then ES is a subset of (x,y), and ES commands U'.

Theorem (25) Suppose that each of X, Y, and Z is a set, X commands Y, and Y commands Z. Then X commands Z.

10.5 Remarks

The following comments contain information about my intent for many of the problems and experiences that my students have had with them.

Definition 1 Notice that the idea of domain is included in the definition of function by proviso ii.

Definition 2, Problem 1, Definition 4 I have written the words for definitions of the sets in question here. Students often translate these to the notation for these words suggested back on the page on sets. If they don't, I usually suggest that they see if they can.

Problems 1 & 3 The students I teach in this course are typically very, very naïve and seldom have been forced to deal with the need to say things carefully. I put Problems 1 & 2 in the notes to address the fact that early in every course, a student would define a "function" as {(x,y) : x is an element of X and y is an element of Y} and then claim whatever additional properties he or she needed as he or she needed them. When it happens now, I can point to Theorem 1 and start the discussion there. Problem 3 reinforces "not every set of ordered pairs whose co-ordinates are in the right sets is a function."

Problem 2 This problem is here to emphasize that, in this set theory, sets have at least one element each. Discussion of the "empty set" will naturally occur here.

Problems 4, 5, 6, & 7 These are all properties assumed about $<$ on $\Re$. Although U will be modified when it fails to yield a density property for precedes, the proofs made in U for 4, 5, & 7 usually go over to the subsequent modification.

Problem 6 This is the first false conjecture since, for instance,
{(x,y) : x is a natural number; and if $x = 1$, then $y = 1$, or if $x > 1$, then $y = 0$}
and,
{(x,y) : x is a natural number; and if $x = 1$, then $y = 0$, or if $x > 1$, then $y = 1$}
have nothing between them. Historically, the resolution to this problem was to identify each such pair as representing a single element. I prefer to emphasize that, in a model, different objects must be different and to continue with a subset of the objects with which we are working.

Definition 7 Commands is the concept central to counting. The Schröder-Bernstein Theorem, which is not addressed in this course, guarantees that it is sufficient to admit the classical results.

Problems 9 & 10 Although all students in your class will know that squaring maps [1,2] onto [1,4], they will not likely realize that this precludes Problem 10 from being a theorem. Indeed, most of the time classes try to prove Problem 10 and the argument includes as its punch line something equivalent to "because the containing set contains more elements than the subset." This affords a marvelous opportunity to clarify the difference between ordinary language and formal language since "more" in the subset sense turns out to be different than "more" in the counting sense (sometimes!). The questions that show incorrect arguments are incorrect often lead to the example $n+1 \to n$.

Problem 11 If Problem 7 has been done at this time, use Problem 11' here instead.

Problem 12 This problem virtually guarantees a case argument will be forthcoming.

Problem 13 Since Problem 6 is not a theorem, if this problem is solved, then it will be done using the order structure. Thus, together with Problem 12, we get that segments defined by an order with neither max nor min create a basis for a Hausdorff topology on the set on which the order is defined.

Definitions 12, 13, and 14 In an axiom system for the numbers, one has (at least) the choice of the greatest lower bound property, the Bolzano-Weierstrass property, the Heine-Borel property, and the Dedekind cut property as a completeness axiom. I choose the Dedekind cut property since it can be articulated without reference to any structure other than the order itself. This is the first idea in the course that the students are likely not to have encountered in another context.

Problems 15 and 16 These problems are usually solved using cuts exhibiting the Dedekind cut property, thus establishing a pretext for asking "must all cuts be like these?"

Problem 17 The order, $<$, as we find it in counting, does not have the Dedekind cut property.

Definitions 4' and 6' and Problems 4'- 8' At some point, Problem 6 will be solved. The example that shows Problem 6 is not a theorem will display two elements of U that have nothing between them. Often students have shown that elements of a particular type do have the property before finding counterexamples, and sometimes the student finding a counterexample will show that any "such pair of elements" fails to have the property. Since the only pairs of elements which fail the property have the "all the rest 0's" and the "all the rest 1's" property, the instructor will have in hand at least an example that makes the structure in Definition 4' plausible. I have placed Problems 4'-8' after Problem 17 *only* because it has been typical in my experience that Problem 6 gets solved before Problems 15-17 get solved. Whenever Problem 6 gets solved, it is time for Definitions 4' and 6' and Problems 4' - 8'. Until Problem 6 gets solved, it is not time for Definitions 4' and 6' or Problems 4' - 8'. An interesting sidelight is to see which constructions from the solutions to 4-8 give elements of U'. These problems offer an outstanding context to point out the power of "some arguments" and always afford at least an opportunity to show how to modify an argument to meet new conditions.

Problems 18 & 19 U' has a countable dense subset.

Problem 4'-7', 18, 19, and 21 collectively demonstrate that U' and G' model the geometry axioms for the numbers and $<$. A class that gets this far has a decent foundation for studying the topology of ordered sets.

Problems 22 & 23 This is the scheme for Cantor's proof that the (place-value model for the) numbers are (is) not denumerable. Some technical care is necessary to ensure that the construction of "x" from 22 is modified to ensure that the object that is made is in U' in order to make it work for 23.

Problem 24 This problem has usually been done in the context of solving 4'-8' (sometimes even earlier), so it may not need to be stated. It is offered here since its construction technique is viable in 25, thus making it a nice lemma for 25.

Problems 26-30 These problems establish that by mimicking something that you can do in U', any set which admits an order with the Dedekind cut property must be non-denumerable. Their inclusion is dependent on whether you wish to concentrate on the model itself (leave them out) or seize the opportunity to illustrate the power of the type of thinking that the students have been doing (include them).

Definitions 17-20 These definitions formalize the place-value addition algorithm. 17 is the digit arithmetic and the carry, 18 creates a notation that will distinguish digit from carry, and 19 manages truncation. 20 matches addition of "terminating decimals" (the ones purged from the system after 6 turned out not to be a theorem) to elements of U'. 19 involves finite induction, so if the issue has not come up before now, here is a chance to teach it.

Problem 32 Although & is "single-valued", it is not an operation on all of U' since any pair of elements which pair 1 with 1 violate the "carry" property (if $\rho((x,y))(1) = pc(((x(1),y(1)),w))$, then Π_1 $pc((((x(1),y(1)),w)) = 0\})$. Students may also find ways to exclude 0 or to exclude 1 from the range of a "prospective answer". This is the first evidence that U' might not be the numbers, since the geometry for the numbers is in place. If the former example is found, the problem on which equations have solutions, 37, is motivated. If the other type of example is found, the plausibility of this model only being the segment from 0 to 1 is established.

Problems 33-35 Wherever it is defined, & has the commutative and associative properties, but does not have an identity. In showing no identity, the solver will likely show that if the function that pairs every natural number with 0, even though it isn't in U', works as an identity when the rule from the algorithm for & is applied.

Problem 36 If 36 is solved before 35, it precludes 35 from being a theorem.

Problems 32' The function &, even though it is not defined on all of U' $\times$ U', is at least single-valued wherever it makes sense.

Problem 38 If 32 is solved by showing the "carry in the first place" is violated, 38 provides an opportunity to show that not having range {0,1} (alternatively, having range {1}) is a possible consequence of applying the algorithm.

What would come next? Using ρ, the algorithm for multiplication can be defined, and the resulting multiplication is an order-reversing, commutative, and associative quasigroup on U' which distributes over & wherever & makes sense. Search for an identity would show that the function that pairs every natural number with 1 would do the job if it were an element of the set on which the algebra acts.

Another tack to take is to extend U' by making element of U" mean element of U', a natural number, or ordered pair whose first co-ordinate is a natural number and whose second co-ordinate is an element of U'. The order is extended lexicographically using $<$ on the natural numbers in the first co-ordinate and G' in the second. The function & is extended by having elements of ($\mathbf{N} \times \mathbf{N}$) $\times$ (U' $\times$ U') paired with pairs whose first co-ordinates are the natural number sums or the natural sums plus 1, depending on whether or not the element of U' $\times$ U' is in E.

10.6 Some Axioms for the Numbers

The primitive words are number, $<$, $+$, and $*$.

Axiom G1 $<$ is an order on the set of numbers.

Axiom G2 It is not the case that the set of numbers has a min by $<$ and it is not the case that the set of numbers has a max by $<$.

Axiom G3 There is a sequence in the set of numbers, call it Q, so that if x and y are numbers, then there is a natural number, call it k, so that x $<$ Q(k) and Q(k) $<$ y.

Axiom G4 $<$ has the Dedekind cut property.

Axiom A1 If each of x and y is a number, then x+y is exactly one number, and $x * y$ is exactly one number.

Axiom A2 If each of x and y is a number, then x+y = y+x, and $x * y = y * x$.

Axiom A3 If each of x, y, and z is a number, then x+(y+z) = (x+y)+z, and $x * (y * z) = (x * y) * z$.

Axiom A4 0 is a number so that if x is a number, then 0+x = x, and 1 is a number so that if x is a number, then $1 * x = x$.

Axiom A5 If x is a number, then there is exactly one number, call it y, so that x+y = 0; and if x is a number different than 0, then there is exactly one number, call it w, so that $x * w = 1$.

Axiom A6 If each of x, y, and z is a number, then $x * (y+z) = (x * y)+(x * z)$.

The Combining Axiom If x and y are numbers so that x < y, and w is a number, then x+w < x+z.

Chapter 11

Topology, Anderson

11.1 Introduction

The introduction is under development.

11.2 Preliminaries

Basic notation and terminology of sets and functions are assumed.

Theorem 1 (DeMorgan's Laws) *Let A and B be subsets of the set X. Then*

1. $X \setminus (A \bigcup B) = (X \setminus A) \bigcap (X \setminus B)$ *and*
2. $X \setminus (A \bigcap B) = (X \setminus A) \bigcup (X \setminus B)$.

Definition 2 *Let I be an indexing set. For each $\delta \in I$ let A_δ be a set. We define the following two sets:*

1. $\bigcup_{\delta \in I} A_\delta = \{x \mid \text{ there exists } \delta \in I \text{ such that } x \in A_\delta\}$ *and*
2. $\bigcap_{\delta \in I} A_\delta = \{x \mid \text{ for all } \delta \in I,\ x \in A_\delta\}$

Theorem 3 (Generalized DeMorgan's Laws) *Let $\{A_\delta \mid \delta \in I\}$ be a collection subsets of the set X. Then*

1. $X \setminus (\bigcup_{\delta \in I} A_\delta) = \bigcap_{\delta \in I} (X \setminus A_\delta)$ *and*
2. $X \setminus (\bigcap_{\delta \in I} A_\delta) = \bigcup_{\delta \in I} (X \setminus A_\delta)$

Theorem 4 *Let A and B be subsets of the set X. Then*

$$A \setminus B = A \bigcap (X \setminus B).$$

Theorem 5 *Let $f\colon X \longrightarrow Y$ be a function and let A and B be subsets of Y. Then*

1. $f^{-1}(A \bigcup B) = f^{-1}(A) \bigcup f^{-1}(B)$,
2. $f^{-1}(A \bigcap B) = f^{-1}(A) \bigcap f^{-1}(B)$,
3. $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$, *and*

4. $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

Theorem 6 *Let $f\colon X \longrightarrow Y$ be a function. Let A be a subset of X and let B be a subset of Y. Then*

1. $A \subset f^{-1}(f(A))$,
2. $f(f^{-1}(B)) \subset B$, *and*
3. $f(X) \setminus f(A) \subset f(X \setminus A)$.

11.3 Theorem Sequence

Definition 7 *A topological space (X, τ) is a set X and a family of sets τ satisfying the following three conditions:*

1. *the empty set, $\emptyset$, and X are members of τ,*
2. *if A and B are in τ, then $A \bigcap B$ is in τ, and*
3. *if I is an indexing set and A_δ is in τ for each δ in I, then $\bigcup_{\delta \in I} A_\delta$ is in τ.*

The members of τ are called open sets and τ is called the topology on X.

Definition 8 *Let (X, τ) be a topological space. A subset A of X is called a closed set if $X \setminus A$ is open.*

Theorem 9 *The union of finitely many closed sets is closed. The intersection of an arbitrary family of closed sets is closed.*

Theorem 10 *For any topological space, (X, τ), the sets $\emptyset$ and X are closed.*

Theorem 11 *Let A be a subset of X. Then A is open if, and only if, for each x in A, there is an open set O_x such that x is a member of O_x and O_x is a subset of A.*

Definition 12 *Let (X, τ) be a topological space and let A be a subset of X. The interior of A, notated* $\operatorname{int}(A)$, *is the union of all open subsets of A. The exterior of A, notated* $\operatorname{ext}(A)$, *is the union of all open sets not intersecting A.*

Theorem 13 *The interior and exterior operators satisfy the following:*

1. $\operatorname{int}(\emptyset) = \emptyset$ *and* $\operatorname{ext}(\emptyset) = X$,

2. $\text{int}(X) = X$ *and* $\text{ext}(X) = \emptyset$,
3. $\text{int}(\text{int}(A)) = \text{int}(A)$,
4. $\text{int}(A \bigcap B) = \text{int}(A) \bigcap \text{int}(B)$,
5. $\text{ext}(A \bigcup B) = \text{ext}(A) \bigcup \text{ext}(B)$,
6. $\text{int}(A) \subseteq A$ *and* $\text{ext}(A) \subseteq X \setminus A$, *and*
7. *if* $A \subseteq B$, *then* $\text{int}(A) \subseteq \text{int}(B)$ *and* $\text{ext}(B) \subseteq \text{ext}(A)$.

Definition 14 *Let x be a member of X and let A be a subset of X. Then x is said to be a boundary point of A if every open set containing x intersects both A and $X \setminus A$. The set of all boundry points of A, notated ∂A, is called the boundry of A.*

Theorem 15 *For every subset A of X, the sets $\text{int}(A)$, $\text{ext}(A)$, and ∂A are mutually disjoint and thier union is X. Moreover, $\text{int}(A)$ and $\text{ext}(A)$ are open sets and ∂A is a closed set.*

Theorem 16 *A set A is closed if, and only if, $\partial A \subseteq A$. A set A is open if, and only if, $\partial A \subseteq X \setminus A$.*

Definition 17 *The closure of a set A, notated $\overline{A}$, is the intersection of all closed sets containing A.*

Theorem 18 *A set A is closed if, and only if, $\overline{A} = a$.*

Theorem 19 *If A is a set, then $\overline{A} = \text{int}A \bigcup \partial A$.*

Theorem 20 *The closure operator satisfies the following:*

1. $\overline{\emptyset} = \emptyset$ *and* $\overline{X} = X$,
2. $A \subseteq \overline{A}$,
3. $\overline{\overline{A}} = \overline{A}$,
4. $\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$, *and*
5. *if* $A \subseteq B$, *then* $\overline{A} \subseteq \overline{B}$.

Theorem 21 *A point x is in $\overline{A}$ if, and only if, every open set containing x intersects A.*

Definition 22 *A point x is a limit point (also called cluster point or accumulation point) of a set A if every open set containing x contains a point of A different from x. The derived set of a set A, notated A', is the set of all limit points of A.*

Theorem 23 *A set A is closed if, and only if, $A' \subseteq A$.*

Theorem 24 *For any set A, $\overline{A} = A \bigcup A'$.*

Definition 25 *Let τ and σ be topologies on X. We say that τ is finer (or larger) than σ if $\sigma \subseteq \tau$. We say that τ is courser (or smaller) than σ if $\tau \subseteq \sigma$. If $\tau \subseteq \sigma$ or $\sigma \subseteq \tau$, then the topologies are said to be comparable. Otherwise, they are not comparable.*

Definition 26 *A family $\mathcal{B}$ of subsets of a set X is a base for a topology on X if the following two conditions are satisfied:*

1. *for each x in X, there is a $B \in \mathcal{B}$ such that $x \in B$ and*
2. *if A and B are in $\mathcal{B}$ and $x \in A \bigcap B$, then there is a C in $\mathcal{B}$ such that $x \in C$ and $C \subseteq A \bigcap B$.*

Theorem 27 *Let $\mathcal{B}$ be a base for a topology on a set X. Let*

$$\tau = \{U |\ U \ is \ the \ union \ of \ members \ of \ \mathcal{B}\}.$$

Then τ is a topology on X.

Definition 28 *The topology τ defined in Theorem 27 is called the topology generated by $\mathcal{B}$.*

Theorem 29 *The topology generated by the base $\mathcal{A}$ is finer than the topology generated by the base $\mathcal{B}$ if, and only if, for any $B \in \mathcal{B}$ and any $x \in B$, there exists an $A \in \mathcal{A}$ such that $x \in A$ and $A \subseteq B$.*

Theorem 30 *A family $\mathcal{B}$ of subsets of X is a base for a given topology τ on X if, and only if, the following two conditions are true:*

1. *for each U in τ and $x \in U$, there is a $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$, and*
2. *$\mathcal{B} \subseteq \tau$.*

Problem 31 (double check this one!) *Let X be a set. Prove or disprove the following statements.*

1. *If I is a set and $\{\tau_\delta |\ \delta \in I\}$ is a collection of topologies on X, then $\bigcap_{\delta \in I} \tau_\delta$ is a topology on X.*
2. *If τ and σ are topologies on X, then $\tau \bigcup \sigma$ is a topology on X.*
3. *If $\{\tau_\delta |\ \delta \in I\}$ is a collection of topologies on X, then there exist unique topologies τ and σ on X such that $\tau \subseteq \tau_\delta \subseteq \sigma$ for all $\delta \in I$.*

Definition 32 *A metric space (X, d) is a set X together with a function $d : X \times X \longrightarrow \mathbf{R}$ which satisfies the following conditions:*

1. *$d(x, y) \geq 0$ for all $x, y \in X$,*
2. *$d(x, y) = 0$ if, and only if, $x = y$,*
3. *$d(x, y) = d(y, x)$ for all $x, y \in X$, and*
4. *$d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.*

The function d is called a metric on X.

Definition 33 *Let (X, d) be a metric space. For $x \in X$ and $r > 0$, the set $B(x, r) = \{y |\ y \in X \text{ and } d(x, y) < r\}$ is called the r-neighborhood (r-ball) about x.*

Theorem 34 *Let (X, d) be a metric space. The collection of all sets $B(x, r)$ such that $x \in X$ and $r > 0$ is a base for a topology on X.*

Definition 35 *The topology generated by r-neighborhoods in Theorem 34 is called the metric topology on X generated by d.*

Definition 36 *A topological space (X, τ) is called metrizable if there is a metric d on X such that the metric topology on X generated generated by d is τ.*

Theorem 37 *Let (X, τ) be a topological space and $Y \subseteq X$. Let $\tau_Y = \{Y \bigcap U |\ U \in \tau\}$. Then τ_Y is a topology on Y.*

Definition 38 *The topological space (Y, τ_Y) in Theorem 37 is called the relative (or induced) topology on Y. Sets in τ_Y are called open in Y or open relative to Y. Similar terminology is used for closed sets.*

Theorem 39 *Let (Y, τ_Y) be a subspace of (X, τ) and $A \subseteq Y$. Then*

1. *A is τ_Y-closed if, and only if, $A = Y \bigcap F$, where F is a τ-closed subset of X,*

2. *a member x of Y is a τ_Y-limit point of A if, and only if, x is a τ-limit point of A,*

3. *the τ_Y-closure of A is the intersection of Y and the τ-closure of A, and*

4. *the intersection of Y and the τ-interior of A is a subset of the τ_Y-interior of A.*

Theorem 40 *Let (Y, τ_Y) be a subspace of (X, τ) and $A \subseteq Y$. Then*

1. *if A is closed in Y and Y is closed in X, then A is closed in X, and*

2. *if A is open in Y and Y is open in X, then A is open in X.*

Definition 41 *A topological space (X, τ) is connected if X is not the union of two nonempty disjoint open sets. A subset Y of X is connected of (Y, τ_Y) is connected.*

Theorem 42 *The space X is connected if, and only if, the only subsets of X which are both open and closed are $\emptyset$ and X.*

Theorem 43 *Let $\{A_\delta |\ \delta \in I\}$ be a collection subsets of the set X. If $A_\alpha \bigcap A_\beta \neq \emptyset$, then $\bigcup_{\delta \in I} A_\delta$ is connected.*

Theorem 44 *Let A be a subset of X. If A is connected and $A \subseteq B \subseteq \overline{A}$, then B is connected.*

Definition 45 *Let A and B be subsets of X. We way that A and B are separated if $A \bigcap \overline{B} = \overline{A} \bigcap B = \emptyset$.*

Theorem 46 *If A and B are both closed or both open, then the sets $A \setminus B$ and $B \setminus A$ are separated sets.*

Theorem 47 *A space X is connected if, and only if, X is not the union of two nonempty separated sets.*

Definition 48 *A nonempty subset C of X is said to be a component of X if*

1. *C is connected, and*

2. *if A is any connected subset of X and $A \bigcap C \neq \emptyset$, then $A \subseteq C$.*

If x is a member of X and C is the component of X such that $x \in C$, then we write $C = C(x)$.

Theorem 49 *Let x be a member of X. Then the component $C(x)$ is the union of all connected subsets of X containing x.*

Theorem 50 *Let (X, τ) be a topological space. Then*

1. *each component of X is closed, and*
2. *if A and B are distinct components of X, then A and B are seperated.*

We will accept the following theorem without proof.

Theorem 51 *The componets of X form a partition of X into maximal connected subsets.*

Definition 52 *A space X is locally connected if it has a basis consisting of connected sets.*

Theorem 53 *If X is locally connected, then the components of open sets are open.*

Theorem 54 *A space X is locally connected if, and only if, for each x in X and each neighborhood U of x, there exists an open connected set V such that $x \in V$ and $V \subseteq U$.*

Definition 55 *Let (X, τ) and (Y, σ) be topological spaces, $f : X \longrightarrow Y$ be a function, and x be a member of X. We say that f is continuous at x if the inverse image of every open set containing $f(x)$ is an open set containing x. That is, f is continuous at x if for each open set V containing $f(x)$, there is an open set U such that $x \in U$ and $f(U) \subseteq V$.*

We say that the function f is continuous if f is continuous at every point in X.

Theorem 56 *Let $f\colon (X, \tau) \longrightarrow (Y, \sigma)$. The following six conditions are equivalent:*

1. *f is continuous,*
2. *the inverse image of each open subset of Y is open in X,*
3. *the inverse image of each closed subset of Y is closed in X,*
4. *the inverse image of each member of a base for σ is open in X,*
5. *for every subset A of X, $f(\overline{A}) \subseteq \overline{f(A)}$, and*
6. *for every subset B of Y, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.*

Definition 57 *A function $F: (X,\tau) \longrightarrow (Y\sigma)$ is a homeomorphism if f is one-to-one (notated 1-1), onto, and both f and f^{-1} are continuous. In this case, (X,τ) and (Y,σ) are said to be topologically equivalent. Any property which when possessed by a space is possessed by all homeomorphic images of that space is called a topological property or a topological invariant.*

Theorem 58 *If X and Y are topological spaces and f is a 1-1 function from X onto Y, then the following are equivalent:*

1. *f is a homeomorphism,*
2. *if G is a subset of X, then $f(G)$ is open in Y if, and only if, G is open in X,*
3. *if F is a subset of X, then $f(F)$ is closed in Y if, and only if, F is closed in X, and*
4. *if E is a subset of X, then $f(\overline{E}) = \overline{f(E)}$.*

Theorem 59 *Let X, Y, and Z be topological spaces, $f: X \longrightarrow Y$, and $g: Y \longrightarrow G$. If f and g are continuous, then $g \circ f\colon X \longrightarrow Z$ is continuous.*

Theorem 60 *If A is a subset of X and $f: X \longrightarrow Y$ is continuous, then $f|_A\colon A \longrightarrow Y$ is continuous.*

Theorem 61 *If $X = A \bigcup B$ where A and B are both open (or both closed) in X and $f\colon X \longrightarrow Y$ is a function such that both $f|_A$ and $f|_B$ are continuous, then f is continuous.*

Theorem 62 *The continuous image of an connected set is connected. That is, if X is connected and $f\colon X \longrightarrow Y$ is continuous, then $f(X)$ is connected.*

Definition 63

1. *A space (X,τ) is called a T_0-space if for each pair of distinct members of X, there is an open set U containing one of the members but not the other.*
2. *A space (X,τ) is called a T_1-space if for each pair of distinct members x and y of X, there is an open set U containing x but not y.*
3. *A space (X,τ) is called a Hausdorff (T_2) space if for each open pair of members x and y in X, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.*
4. *A space (X,τ) is called regular if for each closed subset K of X and x in X, there exist disjoint open sets U and V such that $K \subseteq V$ and $x \in V$.*
5. *A space is called a T_3-space if it is both regular and T_1*

6. *A space (X, τ) is called normal if for each pair, E and F, of disjoint closed subsets of X, there exist disjoint open sets, U and V, such that $E \subseteq U$ and $F \subseteq V$.*

7. *A space is called a T_4-space if it is both normal and T_1.*

Theorem 64 *A space (X, τ) is a T_1-space if, and only if, singleton sets are closed.*

Theorem 65 *If (X, τ) is a Hausdorff space, then*

1. *each finite set is closed, and*

2. *x is a limit point of a subset A of X if, and only if, each open set containing x contains infinitely many members of A.*

Definition 66 *A function $f\colon (X, \tau) \longrightarrow (Y, \sigma)$ is said to be open (resp. closed) if the image of each open (resp. closed) set in X is open (resp. closed) in Y.*

Theorem 67 *If (X, τ) is Hausdorff and $f\colon (X, \tau) \longrightarrow (Y, \sigma)$ is a closed, one-to-one, and onto, then (Y, σ) is Hausdorff.*

Theorem 68 *A space (X, τ) is regular if, and only if, for each x in X and each open set U containing x, there exists an open set V such that $x \in V$ and $\overline{V} \subseteq U$.*

Theorem 69 *A space (X, τ) is normal if, and only if, for each closed set K and open set U containing K, there exists an open set V such that $K \subseteq V \subseteq \overline{V} \subseteq U$.*

Theorem 70 *Every metric space is normal.*

Theorem 71 (Urysohn's Lemma) *A space X is normal if, and only if, for each pair of disjoint closed sets, A and B, in X, there exists a continuous function $f\colon X \longrightarrow [0, 1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.*

Theorem 72 (Tietze's Extension Theorem) *A space X is normal if, and only if, whenever A is a closed subset of X and there is a continuous function $f\colon A \longrightarrow \mathbf{R}$, there exists a continuous extension of f to all of X; that is, there is a continuous function $F : X \longrightarrow \mathbf{R}$ such that $F|_A = f$. Moreover, if f is bounded, then F may chosen to be bounded also.*

Definition 73 *A collection of sets $\Phi = \{A_\delta : \delta \in \mathcal{A}\}$ is called a covering (cover) of X if $X \subseteq \bigcup_{\delta \in \mathcal{A}} A_\delta$.*

Any subcollection of Φ which is also a cover of X is called a subcover.

Definition 74 *A cover Φ of the space X is called an open cover of X if each member of Φ is an open subset of X.*

Definition 75 (We need to think about this one!) *A space (X, τ) is compact if every open cover of X has a finite subcover. A subset Y of X is said to be compact if (Y, τ_Y) is compact.*

Definition 76 *A family of sets $\Phi = \{A_\delta | \, \delta \in \mathcal{A}\}$ has the finite intersection property if the intersection of each finite subfamily of Φ is nonempty; that is, if Ψ is a finite subset of Φ, then $\bigcap_{\delta \in \Psi} A_\delta$ is a nonempty set.*

Theorem 77 *A space X is compact if, and only if, each family of closed subsets of X which has the finite intersection property has a nonempty intersection; that is, if Φ is a collection of closed subsets of X, then $\bigcap_{\delta \in \Phi} A_\delta$ is a nonempty set.*

Theorem 78 *The continuous image of a compact set is compact.*

Theorem 79 *A compact subset of a Hausdorff space is closed.*

Theorem 80 *Disjoint compact subsets of a Hausdorff space have disjoint neighborhoods. That is, if A and B are disjoint compact subsets of a Hausdorff space, then there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

Definition 81 *If X is compact, Y is Hausdorff, and $f \colon X \longrightarrow Y$ is continuous, then f is a closed map.*

Theorem 82 *A one-to-one continuous function from a compact space onto a Hausdorff space is a homeomorphism.*

Definition 83 *A family, φ, of subsets of a set X is a subbase for a topology on X if for each x in X, there is an S in φ such that $x \in S$.*

Theorem 84 *Let φ be a base for a topology on X. Let $\mathcal{B}$ be the set of all finite intersections of members of φ. Then $\mathcal{B}$ is a base for a topology on X.*

Definition 85 *Let Y be a set and let $\{(X_\alpha, \tau_\alpha) | \, \alpha \in A\}$ be a collection of topological spaces. For each α in A, let f_α be a function from Y into X_α. The smallest topology, w, on Y such that for all α in A, $f_\alpha \colon Y \longrightarrow X_\alpha$ is continuous is called the weak topology on Y.*

Theorem 86 *Let Y be a set and let $\{(X_\alpha, \tau_\alpha) | \, \alpha \in A\}$ be a collection of topological spaces. For each α in A, let f_α be a function from Y into X_α. The family $\varphi = \{f^{-1}(U) | U \in T_\alpha$ and $\alpha \in A\}$ is a subbase for the weak topology, w, on Y.*

Definition 87 *1. The Cartesian product of the collection of sets* $\{X_\alpha \,|\, \alpha \in A\}$ *is the set*

$$\prod_{\alpha \in A} X_\alpha = \{x \colon A \longrightarrow \bigcup_{\alpha \in A} \mid x(\alpha) \in X_\alpha \; for \; each \; \alpha \in A\}.$$

2. The set X_α *is called the* α*-th coordinate space and* $x(\alpha)$ *is called the* α*-th coordinate of* x

3. Let $\beta \in A$*. The function* $P_\beta \colon \prod_{\alpha \in A} \longrightarrow X_\beta$ *defined by* $P_\beta(x) = x(\beta)$ *is called the* β*-th projection function.*

4. The product topology on $\prod_{\alpha \in A} X_\alpha$ *is the weak topology determined by the functions* P_α*.*

Theorem 88 *Each projection function is an open function.*

Theorem 89 *A product space is connected if, and only if, each coordinate space is connected.*

Theorem 90 (Alexander's Subbase Theorem) *Let* X *be a topological space and let* φ *be a subbase for the topology on* X*. If every open cover of* X *by members of* φ *has a finite subconver, then* X *is compact.*

Theorem 91 (Tychonoff's Theorem) *A product space is compact if, and only if, each coordinate space is compact.*

Chapter 12

Topology, Mahavier and Mahavier

12.1 To the Instructor

The course sequence that follows started as a draft of my father's notes from a topology course that he taught at Emory University in Atlanta, Georgia for some 30 years. As they came to me, the notes were somewhat rough and adhered to a more traditional style and notation reminiscent of the style of R. L. Moore whose course the notes were based on. In adapting the sequence to my course, I modified, polished, and added to the notes, modernizing the terminology somewhat, but trying to retain the mathematical rigor and precision that was the heart. My students and I have enjoyed the discovery-based format and over time I have learned to cover much more material than I did when I started using this method. The three hour undergraduate topology course follows our Foundations in Mathematics course and serves as a thread of pure mathematics in an otherwise applied curriculum. The next section includes the course syllabus that I pass out to students at the beginning of the semester. This provides insight into the class structure and explains the grading in detail.

I generally pass out the notes to the students a few pages at a time and have them work on the problems. It is understood that the students are to look only to themselves and to me for guidance; no books or outside help of any other kind is to be sought out. Because I sometimes have classes where some of the students have had a semester of real analysis in a similar format, I will often use cooperative learning and pair up the experienced students with one or two inexperienced students. This brings a whole new aspect to the course as they now are working and competing in groups. It also supports the comraderie that we experience as mathematicians when we share our problems.

The course is intended to be a self-contained, one-semester course, although the amount of material covered will vary considerably depending on the experience level of the instructor, the level of the class, and the amount of guidance offered by the instructor on each problem. For a one semester course, I recommend omitting the measure theoretic material and the material dealing with sequence convergence and Cauchy sequences. Omitting these materials will not require altering the sequence in any way. All the theorems can be proved independently of these materials. I often teach the course in this way to concentrate completely on the topological aspects. This allows me to end the course with the development and properties of the Cantor set, the students first introduction to fractals, and to include additional topological spaces in $\Re$, some function spaces, and some metric spaces. For a two semester course one can include all the material and then at the end of this sequence, pick up the real analysis sequence, Section **??**, that includes all the information on continuous functions, differentiability, integrability, etc.

Generally, I offer minimal guidance until the students have nothing to present – then I chat informally about the upcoming definitions, axioms, and theorems so that they have a better intuition. Guidance of this type can easily double the speed of a class and I definitely use this technique to assure that we cover what I consider “sufficient material.” These are *not* lectures, rather they are discussions where students’ questions are turned back to the class for disucssion. The most important aspect of my sucessful classes has been constant open discussion between the students soo that they feel comfortable presenting material at the board, asking questions of myself or students who are presenting, and defending their arguments at the board.

I confess that at the beginning of the semester, I always fear that we are making minimal progress, since students may spend an entire class struggling to put up a simple correct proof, however, by the end of the course, they are putting up two and three correct proofs per class period. The learning curve is exponential and patience is required at the beginning of such a class.

I have received positive feed back from many students who took these courses, but my favorite came from a student who was taking a traditional lecture course in differential equations where the professor proved theorems at the board daily (as he should). The comment was the following: “Finally, I understand what Dr. (blank) puts on the board every day.”

12.2 Course Syllabi

Rules for the course: All work presented or turned in is to be yours or that of your “group.” You are *not* to discuss any problems with any one other than your group (or me), and you are not to look into any books for further guidance. Grading for the course will be the average of three grades: board work grade (group grade), turn in grade (group grade), average of midterm and final exam grades. Anyone who is regularly presenting material at the board will certainly have adequate work for good grades on the written assignments and thus will do well on the midterm and final. I emphasize that the *goal* of the course for each student should be clear presentations of well prepared problems at the board.

Board work: If a problem is about to be presented at the board and you do not wish to see the presentation then you may choose to leave the room. In this case, you may turn in a write up of this problem for credit as original work. You must write *original* at the top of the page. There is no limit on the number of *original* problems you can submit.

Write-ups and originals: You must turn in exactly one “new” problem each week. A “new” problem means one that you have not turned in before. If this problem has been presented in class, label it *write-up.* If it has not been presented, label it *original.* If you receive a grade of less than “B,” you may resubmit this problem on the following week, but you only get one resubmit chance. Please write *resubmit* at the top. Be sure that everything you turn in is double-spaced with your name, problem number, and problem statement on it. Be sure and write either *write-up, original,* or *resubmit*, at the top of each problem turned in.

Grading for turn in assignments and board work will be based on the following scale.

- A, This is a correct proof.
- B, I believe you know how to prove the theorem but some of what you have written is not correct.
- C, I cannot tell if you understand the problem based on what you have written.
- D, There is at least one major flaw in your argument.

Please understand that the purpose of these exercises is to *teach* you to prove theorems, it is not expected that you started the class with this knowledge; hence, some low grades are to be expected. Do not be upset – just come see me.

12.3 To the Student

Topology is an area of mathematics, just as Algebra, Analysis and Geometry are areas of mathematics. Like other areas Topology is generally defined heuristically or not at all. The kind of problems we shall consider first are those that have to do with the concept of a limit point of a set of real numbers or the limit of a convergent sequence of real numbers as defined in an Analysis course such as Calculus. There the definition is made in terms of the distance between points and involves the concept of numbers being "near" one another. Precisely, the number x is a limit point of the set M of numbers if for every positive number, ϵ, there is a point of M which is different from x and whose distance from x is less than ϵ. We start by defining a limit in a more abstract setting. We do this by introducing a notion of "nearness" which does not depend on having a distance between points. So we might consider this part of topology the study of those concepts with can be defined in terms of limit points.

While topological spaces can be defined in very general settings, this sequence is restricted to the study of linear topology, that is the study of the topological properties of the real line. Many of the results that we will prove for the line hold in general topological spaces and often the proofs given in class will not use any properties of the line and thus are actually proofs for general topological spaces. In brief, our goal is an understanding of the following topics: open and closed sets, limit points, compactness, connectedness, measure of a set, sequences, convergence, infemum, supremum, etc. While investigating these topics, we will be developing the tools that are needed for such courses as general topology, measure theory, functional analysis, differential equations, and so forth.

12.4 Theorem Sequence

Definition 1 *By a* **point** *is meant an element of the real numbers,* $\Re$.

Definition 2 *By a* **point set** *is meant a collection of one or more points.*

Definition 3 *The statement that the set* S *is a* **topological space** *means that there is a collection of subsets of* S*, called regions, such that*
i) if p *is in* S *then there is a region that contains* p*, and*
ii) if U *and* V *are two regions having* p *in common then there is a region which contains* p *and is a subset of* $U \cap V$.

Definition 4 *The statement that the point set* M *is* **linearly ordered** *means that there is a meaning for the words "less than* ($<$)*," "less than or equal to* ($\leq$)*," "greater than* ($>$)*," and "greater than or equal to* ($\geq$)*." If each of* a*,* b *and* c *is in* M*, then*

- *if* $a \leq b$ *and* $b \leq c$ *then* $a \leq c$
- *one and only one of the following is true:*
  - $a \leq b$,
  - $b \leq a$, *or*
  - $a = b$.

Axiom 5 $\Re$ *is linearly ordered.*

Axiom 6 *If* p *is a point there is a point less than* p *and a point greater than* p.

Axiom 7 *If* p *and* q *are two points then there is a point between them, for example,* $(p+q)/2$.

Axiom 8 *If* $a < b$ *and* c *is any point, then* $a + c < b + c$,

Axiom 9 *If* $a < b$ *and* $c > 0$*, then* $a \cdot c < b \cdot c$*. If* $c < 0$*, then* $a \cdot c > b \cdot c$.

Axiom 10 *If* x *is a point, then* x *is an integer or there is an integer* n *such that* $n < x < n + 1$.

Definition 11 *The statement that the point set* O *is an* **open interval** *means that there are two points* a *and* b *such that* O *is the set of all points between* a *and* b.

Definition 12 *The statement that I is a* **closed interval** *means that there are two points a and b such that $p \in I$ if and only if p=a, p=b, or p is between a and b.*

Notation: We use the notation (a,b) to denote the open interval consisting of all points p such that $a < p < b$. Similarly we use the notation [a,b] to denote the closed interval determined by the two points a and b where $a < b$. We do not use (a,b) or [a,b] in case $a = b$, although many mathematicians and texts do.

Definition 13 *If M is a point set and p is a point, the statement that p is a* **limit point** *of M means that every region containing p contains a point of M different from p.*

Problem 14 *Determine if $\Re$ is a topological space if regions are defined to be sets contaning exactly one point. I.e. R is a region if and only if $R = \{p\}$ for some $p \in \Re$.*

Problem 15 *Determine if $\Re$ is a topological space if the only region is the entire space, $\Re$.*

Problem 16 *Determine if $\Re$ is a topological space if only closed intervals are regions.*

Problem 17 *Determine if $\Re$ is a topological space if only half open intervals, open on the right, are regions. That is, R is a region if and only if there are numbers a and b with $a < b$ such that $R = \{x | a \leq x < b\}$. Hint: If this were a topological space, then it would be referred to as the* **Sorgenfrey line**.

Theorem 18 *Prove that $\Re$ is a topological space if only open intervals are regions.*

Note: From this point on we interpret $\Re$ to mean the topological space where regions are defined to be open intervals. This topolgoical space would be referred to as the **usual topology on** $\Re$ or the **Euclidean topology on** $\Re$.

Definition 19 *The statement that the set S is a* **Hausdorff** *space means that S is a topological space and if p and q are two (distinct) elements of S then there are mutually disjoint regions U and V containing p and q respectively.*

Theorem 20 $\Re$ *is a Hausdorff space.*

Definition 21 *The statement that the sequence* $p_1, p_2, p_3, \ldots$, *denoted* (p_i), **converges to the point p** *means that if* R *is a region containing* p, *then there is a positive integer* n *such that if* m *is a positive integer and* $m > n$, *then* p_m *is in* R.

Definition 22 *The statement that the sequence* (p_i) **converges**, *means that there is a point* p *such that* (p_i) *converges to* p.

Problem 23 *For each positive integer* n, *let* $p_n = 1 - 1/n$. *Show that the sequence* (p_i) *converges to* 1.

Problem 24 *If* m *is a positive, odd integer then* $p_m = 1/m$ *while if* m *is a positive, even integer then* $p_m = (m+1)/m$. *Show that the sequence,* (p_i) *does not converge to zero.*

Problem 25 *For each positive integer* n, *let* $p_{2n} = 1/(2n-1)$, *and let* $p_{2n-1} = 1/2n$. *Show that the sequence* (p_i) *converges to* 0.

Definition 26 *If* M *is a point set, then the closure of* M, *denoted by* **Cl(M)**, *is the set to which the point* p *belongs if and only if* p *is a point of* M *or* p *is a limit point of* M.

Definition 27 *The statement that the topological space* S *is* **regular at the point p** *means that if* U *is a region containing* p, *there is a region* V *containing* p *such that* $Cl(V) \subseteq U$.

Definition 28 *The statement that the topological space* S *is* **regular** *means that* S *is regular at each of its points.*

Theorem 29 $\Re$ *is a regular space.*

Definition 30 *If* (p_i) *is a sequence, then the set* $\{p_i : i \text{ is a positive integer }\}$ *denotes the* **range** *of the sequence. That is,* $\{p_i : i \text{ is a positive integer }\}$ *denotes the point set to which the point* p *belongs if and only if there is a positive integer* n *such that* $p = p_n$.

Problem 31 *Show that if the sequence* (p_i) *converges to the point* p, *and, for each positive integer* n, $p_n \neq p_{n+1}$, *then* p *is a limit point of the set which is the range of the sequence.*

Definition 32 *The statement that the set* M *is* **finite** *means that there is a positive integer,* n, *such that* M *has* n *points and does not have* n+1 *points.*

Definition 33 *The statement that the set M is* **infinite** *means that M is not finite.*

Definition 34 *A* **rightmost point** *of a point set M is a point, r, such that $r \in M$ and $r \geq m$ for all $m \in M$.* **Leftmost** *is defined analogously.*

Definition 35 *A* **first point to the right** *of a point set M is a point, r, such that $r > m$ for all $m \in M$ and there is no point s satisfying $s < r$ and $s > m$ for all $m \in M$.* **First point to the left of M** *is defined analogously.*

Theorem 36 *If M is a finite point set then M has a leftmost and rightmost point.*

Theorem 37 *If p is a limit point of the point set M, then every region containing p contains infinitely many points of M.*

Problem 38 *Show that if c is a number and (p_i) is a sequence which converges to the point p, then the sequence (cp_i) converges to cp.*

Problem 39 *Show that if the sequence (p_i) converges to p and the sequence (q_i) converges to q, then the sequence $(p_i + q_i)$ converges to $p + q$.*

Definition 40 *The statement that the point sets H and K in a topological space are* **mutually separated** *means that neither contains a point nor a limit point of the other.*

Definition 41 *The statement that the point set M in a topological space S is* **connected** *means that M is not the union of two mutually separated sets.*

Theorem 42 *If p is a limit point of the point set H and H is a subset of the point set K, then p is a limit point of K.*

Theorem 43 *If H and K are point sets and p is a limit point of $H \cup K$, then p is a limit point of H or p is a limit point of K.*

Theorem 44 *If a connected set M is a subset of the union of two mutually separated point sets H and K, then it is a subset of one of them.*

Theorem 45 *If the sequence (p_i) converges to the point p and q is a point different from p, then (p_i) does not converge to q.*

Theorem 46 *If the sequence (p_i) converges to the point p and q is a point different from p, then q is not a limit point of the range of the sequence (p_i).*

Definition 47 *The statement that the point set M is* **open** *means that if p is a point of M, then there is a region, R, satisfying, $p \in R \subseteq M$.*

Note: Every region is an open set. Can you find some other open sets?

Definition 48 *The statement that the point set M is* **closed** *means that if p is a limit point of M, then $p \in M$.*

Theorem 49 *If H and K are closed point sets then $H \cup K$ and $H \cap K$ are closed.*

Theorem 50 *If H and K are regions then $H \cup K$ and $H \cap K$ are open.*

Definition 51 *If S is a set and M is a proper subset of S then M^c is defined by $M^c = \{s \in S : s \notin M\}$.*

Theorem 52 *If M is a point set and M is closed, then M^c is open.*

Theorem 53 *If M is a point set and M is open, then M^c is closed.*

Theorem 54 *If $\mathcal{G}$ is a finite collection of regions, each containing the point p, then the set of all points which are in all the sets in $\mathcal{G}$ is an open point set.*

Theorem 55 *If $\mathcal{G}$ is an arbitrary (i.e. possibly infinite) collection of closed point sets, each containing the point p, then the set of all points which are in all the sets in $\mathcal{G}$ is a closed point set.*

Theorem 56 *If $\mathcal{G}$ is a finite collection of closed point sets, then $\mathcal{G}^*$ is closed.*

Theorem 57 *If $\mathcal{G}$ is an arbitrary collection of open sets, then $\mathcal{G}^*$ is open.*

Note: If $\mathcal{G}$ is a collection of point sets, then the union of the members of $\mathcal{G}$ is denoted by $\cup\{G|G \in \text{G}\}$ or $\cup_{G\in\mathcal{G}}G$ or, more simply, by $\mathcal{G}^*$. Similarly, the set of points common to the members of $\mathcal{G}$, called the intersection of the members of $\mathcal{G}$ is denoted by $\cap\{G|G \in \text{G}\}$ or $\cap_{G\in\mathcal{G}}G$.

The previous few theorems imply:

- The collection of open sets is closed under the operation of arbitrary union.
- The collection of open sets is closed under the operation of finite intersection.

- The collection of closed sets is closed under the operation of arbitrary intersection.
- The collection of closed sets is closed under the operation of finite union.

Note: Some people would use these facts as the definition of a topological space. They would say that a **topological space** was an ordered pair, (S, τ) where S is the overlying space and τ is a collection of subsets of S, called open sets, such that
i) if p is a point in S then there is a member of τ that contains p,
ii) τ is closed under the operation of finite intersection
iii) τ is closed under the operation of arbitrary union.

If we were using this definition for a topological space, then the defnition that we are currently using would be referred to as a basis for the topological space. Thus, using our definition, we obtain all the open sets by taking all possible finite intersections and all possible infinite unions of collections of regions.

Theorem 58 *If H and K are two mutually disjoint closed point sets, they are mutually separated.*

Theorem 59 *If H and K are connected point sets having a point p in common, then* $H \cup K$ *is connected.*

Theorem 60 *If H is a connected point set and K is a point set and every point of K is a limit point of H, then* $H \cup K$ *is connected.*

Definition 61 *The statement that the point set M is* **bounded above** *means that there is a point to the right of every number in M. The statement that M is* **bounded below** *is defined similarly.*

Definition 62 *M is* **bounded** *means that M is bounded above and bounded below.*

Theorem 63 *If the sequence* (p_i) *converges to the point p, then the range of this sequence is bounded.*

Definition 64 *The statement that the sequence* (p_i) *is an* **increasing** *sequence means that for each positive integer n,* $p_n < p_{n+1}$.

Definition 65 *The statement that the sequence* (p_i) *is* **non-decreasing** *means that for each positive integer n,* $p_n \leq p_{n+1}$. *We define a* **decreasing** *and* **non-increasing** *sequence similarly.*

Theorem 66 *If* (p_i) *is a non-decreasing sequence and there is a point to the right of each point of the sequence, then the sequence converges to some point.*

Theorem 67 *If G is a collection of connected point sets and one of them intersects all the others, then* G^* *is connected.*

Definition 68 *If M is a point set in a topological space S, then by a* **component** *of M is meant a connected subset of M that is not a subset of any other connected subset of M.*

Theorem 69 *If M is a point set and p is a point of M, then there is exactly one component of M which contains p.*

Definition 70 *The statement that the sequence* (p_i) *is a* **Cauchy sequence** *means that if* ϵ *is a positive number, then there is a positive integer n such that if m is a positive integer and k is a positive integer,* $m \geq n$*, and* $k \geq n$*, then the distance from* p_m *to* p_k *is less than* ϵ*.*

Note: More formally, but equivalently, the statement that the sequence (p_i) is a Cauchy sequence means that if ϵ is a positive number then there is a positive integer N such that if each of m and n is an integer, $m > N$ and $n > N$ then $|p_m - p_n| < \epsilon$

Note: Any theorems which use the definition of a Cauchy sequence assume, in addition to our other axioms, that we have a distance between the points in our space. On the number line, but the distance from the point a to the point b is meant $|a - b|$. Recall that $|p|$ is p or -p according as p is non-negative or negative.

Theorem 71 *If* (p_i) *is a sequence converging to the point p, then the sequence* $p_1 - p_2, p_2 - p_3, \ldots$ *converges to 0.*

Note: The converse to Theorem 14 is, surprisingly, false. Find an example to show this.

Axiom 72 *If M is a point set and there is a point to the right of every point of M, then M has either a rightmost point or a first point to the right.*

Definition 73 *The point described in axiom 72 is called the* **least upper bound** *of M or the* **supremum** *of M. The point which would be described if left and right were interchanged is called the* **greatest lower bound** *of M or the* **infemum** *of M. These are usually denoted by* **lub(M)** *or* **sup(M)** *and* **glb(M)** *or* **inf(M)**.

Note: We have already been discussing the conecpts of lub(M), but calling it either the rightmost point of M or the first point to the right of M, depending on whether the point was an element of M or not. Axiom 72 is usually called the **Completeness axiom**. Other equivalent axioms are **Zorn's lemma, the Hausdorf maximal principle, Tukey's lemma, Zermelo's theorem, Tychonoff's theorem,** and the **axiom of choice.** A linearly ordered space which satisfies any of the equivalent forms would be called a **complete** space. Thus at this point we could say we have a **complete, linearly ordered space which has no minimal or maximal element.** I will mark with (AC) those theorems which require the completeness axiom. We assume of course that a statement similar to axiom 72, but with right and left reversed also holds. We might restate axiom 72 as follow: "If M is bounded above (below) then M has a least upper bound (greatest lower bound)."

Theorem 74 *If M is a point set, there is a point to the right of every point of M, b is the least upper bound of M and b is not in M, then b is a limit point of M.*

Theorem 75 *(p_i) is a Cauchy sequence if and only if it is true that for each positive number d, there is a positive integer n such that if m is a positive integer and $m \geq n$, then $|p_m - p| < d$.*

Theorem 76 *If the sequence (p_i) converges to a point p, then (p_i) is a Cauchy sequence.*

Theorem 77 *If (p_i) is a Cauchy sequence, then the range of (p_i) is bounded.*

Theorem 78 *If H and K are bounded sets and $H \subseteq K$ then $lub(H) \leq lub(K)$ and $glb(H) \geq glb(K)$.*

Theorem 79 *If H and K are bounded sets and L is the set to which the number x belongs if and only if there is a number $h \in H$ and a number $k \in K$ such that $x = h + k$, then $glb(H) + glb(K) = glb(L)$ and $lub(H) + lub(K) = lub(L)$.*

Theorem 80 *If (p_i) is a Cauchy sequence, then the range of (p_i) does not have two limit points.*

Theorem 81 *If (p_i) is a Cauchy sequence, then the sequence (p_i) converges.*

Definition 82 *If (a,b) is a segment, then by the* **length** *of (a,b) is meant b-a.*

Definition 83 *If $\mathcal{G}$ is a collection of segments, let* $\mathbf{L}(\mathcal{G})$ *denote the set of all numbers which are the sums of the lengths of finite subsets of $\mathcal{G}$.*

Problem 84 *Show that if $\mathcal{G}$ is a finite collection of segments, then lub($L(\mathcal{G})$) is the sum of the lengths of the segments in $\mathcal{G}$.*

Problem 85 *Show that if $\mathcal{G}$ is the summable collection of segments defined by $\mathcal{G} = \{(a_i, b_i)\}_{i=1}^{\infty}$ then the sequence $\{\sum_{i=1}^{j}(b_i - a_i)\}_{j=1}^{\infty}$ converges to the $lub(L(\mathcal{G}))$.*

Definition 86 *If $\mathcal{G}$ is a summable collection of segments then by the* **sum of the lengths** *of the members of $\mathcal{G}$ is meant lub($L(\mathcal{G})$).*

Definition 87 *The statement that $\mathcal{G}$ is a* **summable** *collection of segments means that $\mathcal{G}$ is a collection of segments such that $L(\mathcal{G})$ is bounded.*

Theorem 88 *If (p_i) is a sequence of points in the closed interval $[a, b]$, then there is a point in $[a, b]$ which is not in the sequence (p_i).*

Theorem 89 *If p is a limit point of the point set M then there is a sequence of points $p_1, p_2, p_3, \ldots$ of M, all different and none equal to p which converge to p.*

Theorem 90 *If (p_i) is a sequence of distinct points in the closed interval, $[a, b]$ then the range of the sequence has a limit point.*

Definition 91 *The statement that the collection $\mathcal{G}$ of point sets* **covers** *the set K means that if p is a point of K, then there is an element $g \in \mathcal{G}$ such that $p \in g$. We call G a* **cover** *of K.*

Problem 92 *Find a collection $\mathcal{G}$ of open intervals covering the open interval (a,b) such that no finite subset of $\mathcal{G}$ covers (a,b)*

Problem 93 *Find a collection $\mathcal{G}$ of closed intervals covering the open interval (a,b) such that no finite subset of $\mathcal{G}$ covers (a,b).*

Problem 94 *Find a collection $\mathcal{G}$ of closed intervals covering the closed interval [a,b] such that no finite subset of $\mathcal{G}$ covers [a,b].*

Theorem 95 *(AC) If $\mathcal{G}$ is a collection of open intervals covering the closed interval [a,b], then some finite subset of $\mathcal{G}$ covers [a,b].*

Definition 96 *If $L = \{ L(G) : G$ is a collection of open intervals covering $M \}$ then the* **outer measure** *of a point set, M, is defined by m(M) = glb(L).*

Problem 97 *Show that the outer measure of the open interval, (a,b), and the closed interval, [a,b], is b-a.*

Problem 98 *Show that if M is a finite set, M has outer measure zero.*

Definition 99 *The statement that the point set M is* **countable** *means that M is finite or there is a sequence $(p_i)_{i=1}^{\infty}$ of distinct points such that for each point x in M, there is a positive integer i such that $p_i = x$.*

Problem 100 *Show that if M is a countable point set then M has outer measure zero.*

Problem 101 *Find a sequence $S_1, S_2, S_3, \ldots$ of open intervals such that for each positive integer n, $S_{n+1} \subseteq S_n$, and*

a) $\cap_{n=1}^{n=\infty} S_n$ is an open interval

b) $\cap_{n=1}^{n=\infty} S_n$ is not an open interval.

Problem 102 *Find a sequence $I_1, I_2, I_3, \ldots$ of closed intervals such that for each positive integer n, $I_{n+1} \subseteq I_n$ and*

a) $\cap_{n=1}^{n=\infty} I_n$ is a closed interval

b) $\cap_{n=1}^{n=\infty} I_n$ is not a closed interval.

Problem 103 *Find an example of a sequence of closed point sets, each containing the next such that there is no point common to all the sets of the sequence.*

Lemma 104 *(AC) If $\{M_i\}_{i=1}^{\infty}$ is a nested sequence of closed intervals then $\cap_{i=1}^{\infty} M_i$ is either a single point or a closed interval.*

Lemma 105 *If $\{M_i\}_{i=1}^{\infty}$ is a nested sequence of closed sets and there is a point common to all of them and p is a limit point of the set of all such points then p is a limit point of M_k for all k=1,2,...*

Theorem 106 *If $\{M_i\}_{i=1}^{\infty}$ is a sequence of closed point sets and there is a point common to all the sets of the sequence $\{M_i\}_{i=1}^{\infty}$, then the set of all such points is a closed point set.*

Definition 107 *The statement that the point set M is* **conditionally compact** *means that if K is an infinite subset of M, then K has a limit point.*

Problem 108 *(AC) Show that the segment, (a,b), is conditionally compact.*

Problem 109 *Find an example of a sequence of conditionally compact sets, each containing the next such that there is no point common to all the sets of the sequence.*

Lemma 110 *Show that if M is conditionally compact then M is bounded.*

Theorem 111 *If $\{M_i\}_{i=1}^{\infty}$ is a sequence of closed and conditionally compact point sets such that for each positive integer i, $M_{i+1} \subseteq M_i$, then there is a point common to all the sets of the sequence $\{M_i\}_{i=1}^{\infty}$ and the set of all such points is a closed and conditionally compact point set.*

Definition 112 *The statement that the point set M is* **compact** *means that if $\mathcal{G}$ is a collection of regions covering M, then some finite subset of $\mathcal{G}$ covers M.*

Problem 113 *Find an example of a point set which is closed, bounded, and every point of M is a limit point of M, and which is not an interval.*

Theorem 114 *(AC) If M is an infinite and bounded point set then M has a limit point.*

Note: This theorem along with lemma 110 imply that a set M is conditionally compact if and only if it is bounded.

Lemma 115 *Show that if M is a countable set then there exists a nested sequence of closed intervals $I_1, I_2, I_3, \ldots$ such that $\cap_{i=1}^{\infty} I_k$ contains no point of M.*

Theorem 116 *If (p_i) is a sequence of distinct points in the closed interval [a,b], then there is a point in [a,b] which is not in the sequence.*

Definition 117 *The statement that the set K is* **dense in the set M** *means that every point of M is a point or a limit point of K.*

Note: This definition would probably most often be given by saying K is dense in M means that $Cl(K) = M$. These are equivalent.

Theorem 118 *There is a sequence (p_i) of distinct points in the interval [a,b] such that the range of the sequence is dense in [a,b].*

Problem 119 *Find a set which is closed, bounded, every point is a limit point and contains no interval.*

Theorem 120 *No countable and closed point set M has the property that every point of M is a limit point of M.*

Note: This theorem guarantees that the Cantor set described above is not countable. Can you find a rational number that is in the Cantor set? Can you find a point of the Cantor set that is not an endpoint of one of the intervals used in the construction of the set? Can you find an irrational number that is in the Cantor set?

Theorem 121 *If M is a countable subset of an interval [a,b] then every point of M is a limit point of [a,b]-M.*

Note: It follows from the previous theorem that the set of all irrational numbers in the interval [a,b] is dense in the interval [a,b].

Chapter 13

Trigonometry, W. T. Mahavier

13.1 To the Instructor

The following sequence was developed for use in a one semester trigonometry course at a small, regional, open-enrollment college. The course came to life because over the past ten years I have felt less and less comfortable with the trigonometry courses that I taught at Emory, The University of North Texas, and Nicholls State University. I constantly found myself veering away from the recommended textbook because of a lack of rigor. The emphasis of the texts was on the quantity and type of problems rather than on the precision of the material and the development of problem solving skills in the students. This sequence is my attempt to produce in the students an understanding of the trigonometric functions while honing their communication, presentation, and problem solving skills. The typical class size is about ten students. For larger classes, which I have yet to encounter, I would expect to have the students work in groups where one person represents the group for the presentation of a given problem, a technique that Charles Allen and Carol Browning have had success with at Drury University.

The structure of the class is simple. I pass the notes to the students a few pages at a time and have them work on and present the problems. It is understood that they are to look only to themselves and to me for guidance; no books or outside help of any other kind is to be sought out. As a problem is presented, I turn to the class and ask if it is correct or if there are questions. While I might lead with questions to the audience, I rarely point out mistakes at the board and by midterm I have placed the burden of determining the correctness of each problem completely on the class. If I am asked if a problem is correct, I merely take a vote from the class or ask what they are worried about. I often jest that if there is a mistake then surely they will find it if I put the problem on the final or I suggest that we take a vote at the beginning of class the next day.

Generally, I offer minimal guidance until the students have nothing to present. Then I chat informally about the upcoming definitions and problems so that they have a better intuition. Direction of this type can easily double the speed of a class and I definitely use this technique to assure that we cover what I consider "sufficient material." These are *not* lectures, rather they are discussions where students' questions are turned back to the class for debate. The most important aspect of my sucessful classes has been constant open discourse between the students so that they feel comfortable presenting material at the board, asking questions of myself or students who are presenting, and defending their arguments at the board.

At the beginning of the semester, I always fear that we are making minimal progress, since students may spend an entire class struggling to put up a simple problem, however, by the end of the course, they are putting up many correct problems per class period. The learning curve is exponential and the patience required at the beginning is rewarded as they learn to read carefully and do the mathematics on their own. The approach and demeanor of the instructor is the critical element for success. The second time I taught this class, I added about 30% to the notes and we covered all the notes as they are presented here.

13.2 To the Student

The structure of this course will quite likely be different from previous courses you have taken. There will be no book and all the notes that you will need will be provided. These notes and my office hours are to be your only resources. The notes that you will develop as you work through the problems in this sequence will be your book, a collection of problems and solutions that you and your peers, rather than me or the author of some text book, have worked out. The purpose of this format is to actively involve you in the process of *doing* mathematics as opposed to passively viewing a lecture and then mimicking problems slightly modified from those you have been shown. The bad news is that this approach is very different from your previous classes. The good news is that it is a lot more fun to participate in a sport than to watch it on TV.

We will derive the subject of trigonometry essentially from scratch. Therefore, when you work a problem you are not allowed to use anything that we have not already discussed in the course with the exception of a few concepts listed in the next paragraph or any concepts that you may create or define on your own. You will not be allowed to use information that you may recall from previous courses such as the fact that $sin(\theta) = o/h$ until we have defined these terms and proved this fact based on material developed in this course.

On the other hand, there are concepts that we will assume are familiar and you should feel free to use as needed. We might refer to these as our "undefined terms" since we will use them without stating a formal definition.

- circle, coordinate axes, origin, point
- line, line segment, ray
- arc, area, center, circumference, diameter, radius, and tangent of a circle
- triangle, similar triangles, square

All work presented or turned in is to be yours or that of your group. You are *not* to discuss any problems with any one other than your group (or me), and you are not to look into any books for further guidance. Grading for the course will be the average of three grades: (i) the average of your board work grades (group grades), (ii) the average of your written assignments, and (iii) the average of midterm and final exam grades. Anyone who is regularly presenting material at the board will certainly have adequate work for good grades on the written assignments and thus will do well on the midterm and final. I emphasize that the *goal* of the course for each student should be clear presentations at the board of well prepared problems.

If a problem is about to be presented at the board and you do not wish to see the presentation then you may choose to leave the room. In this case, you may turn in a write up of this problem for credit as original work. You must write *original* at the top of the page. There is no limit on the number of *original* problems you can submit.

You must turn in exactly one *new* problem each week. A *new* problem means one that you have not turned in before. If this problem has been presented in class, label it *write-up.* If it has not been presented, label it *original.* If you receive a grade of less than "B," you may resubmit this problem on the following week and I will record the higer of the two grades. You only get one chance to resubmit. Please write *resubmit* at the top. Be sure that everything you turn in is double-spaced with your name, problem number, and problem statement on it. Be sure and write either *write-up, original,* or *resubmit*, at the top of each problem turned in.

Grading for turn in assignments and board work will be based on the following scale.

- A, This is a correct problem.
- B, I believe you know how to do the problem but some of what you have written is not correct.
- C, I cannot tell if you understand the problem based on what you have written.
- D, There is at least one major flaw in your argument.

The purpose of these exercises is to *teach* you to solve problems and write the solutions correctly. It is not expected that you started the class with this knowledge; hence, some low grades are to be expected. Do not be upset – just come see me and resubmit the problem.

13.3 Problem Sequence

Definition 1 *The* **unit circle** *is the circle centered at the origin of radius one.*

Problem 2 *Graph the unit circle and subdivide it into eight arcs of equal length such that one division lies at the point (1,0). Working in a counter-clockwise direction, determine the distance along the unit circle from the point (1,0) to each of the divisions and label each division with this distance.*

Problem 3 *Repeat the previous exercise using a total of twelve divisions.*

Problem 4 *Repeat the previous two exercises using a circle centered at the origin and of radius two.*

Definition 5 *An* **angle** *is a subset of the plane consisting of two distinct rays (or two distinct line segments) with a common endpoint called the* **vertex.**

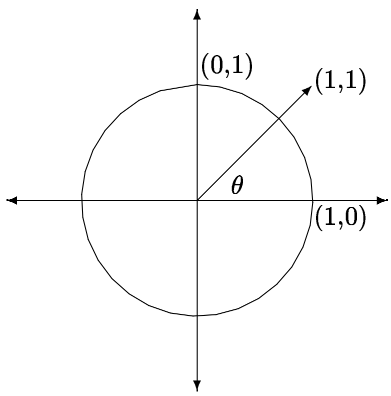
Definition 6 *An* **angle in standard position** *is an angle where one of the two rays is the postive x-axis. This ray is referred to as the* **initial side** *of the angle. The other ray is referred to as the* **terminal side** *of the angle.*

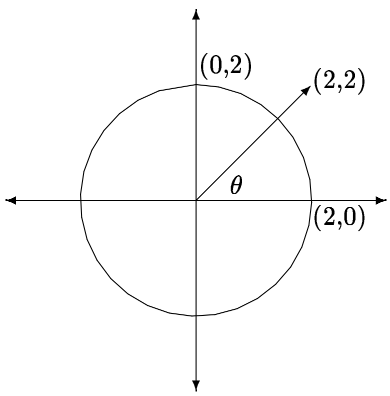
Definition 7 *Given a circle centered at the origin and an angle in standard position, let P_1 be the intersection of the circle with the initial side of the angle and let P_2 be the intersection of the circle with the terminal side of the angle. The* **arc associated with the angle** *is the portion of the circle traced by a point traversing the circle in a counter clockwise direction from the point P_1 to the point P_2.*

Definition 8 *Given a circle centered at the origin and an angle in standard position the* **radian measure of the angle** *is the ratio of the length of the arc associated with the angle to the radius of the circle.*

Notation: If the radian measure of an angle is θ then we will say that such an angle has measure θ radians.

Problem 9 *Determine the radian measure of each angle illustrated below.*





Problem 10 *Given an angle, not necessarily in standard position, make up a definition for the* **radian measure of the angle**.

Definition 11 *Given a circle centered at the origin and an angle in standard position the* **degree measure of the angle** *is $360/2\pi$ times the radian measure of the angle.*

Historically, the circle was evenly divided into 360 arcs and an angle was said to have degree measure θ if the terminal side of the angle intersected the circle at the θ^{th} division. I have read two explanations of this – the choice of 360 was because it was believed there were 360 days in a year or because much mathematics was done based on multiples of 60. If pressed, I could probably come up with a source supporting these statements.

Definition 12 *A* **right triangle** *is a triangle that has one angle with degree measure 90.*

Theorem 13 *The sum of the degree measures of the interior angles of a triangle is 90.*

Definition 14 *The* **hypotenuse** *of a right triangle is the side that is not adjacent to the angle of degree measure 90.*

Theorem 15 *Pythagorean Theorem: Given a right triangle with sides of length, a, b, and c where c is the length of the hypotenuse, we have $a^2 + b^2 = c^2$.*

Notation: If the degree measure of an angle is θ then we will say that such an angle has measure θ degrees.

Problem 16 *Determine the degree measure of each angle in the previous illustration.*

Problem 17 *Graph the unit circle and the angle in standard position whose measure is 150 degrees.*

Problem 18 *Graph the unit circle and the angle in standard position that has measure $\pi/4$ radians. What are the x-y-coordinates of the point that is the intersection of the terminal side of this angle with the unit circle?*

Problem 19 *Graph the unit circle and the angle in standard position that has measure $\pi/3$ radians. What are the x-y-coordinates of the point that is the intersection of the terminal side of this angle with the unit circle? What is the length of the arc associated with this angle?*

Problem 20 *Graph a circle of radius two centered at the origin and the angle in standard position so that the length of the arc associated with the angle is $3\pi/2$. What is the radian measure of this angle? What is the degree measure of this angle?*

Problem 21 *Given a circle centered at the origin with radius five centimeters and an angle in standard position with radian measure $7\pi/4$, determine the length of the arc associated with this angle.*

Problem 22 *Suppose that a unicycle with a wheel of radius 9 inches is rolled four feet. Through what radian measure has one spoke on this wheel traveled? How many revolutions has the wheel made?*

Problem 23 *Suppose that it took 2 seconds to roll the unicycle (described in the previous problem) 2 feet. What is the speed of the unicycle as measured in inches per second? As measured in radians per second? As measured in revolutions per second?*

Problem 24 *Given an angle of degree measure 270, determine the radian measure of the angle.*

Problem 25 *Given an angle of radian measure* $2\pi/3$, *determine the degree measure of this angle.*

The development above hinges on the choice of the *counter-clockwise* direction in the definition of the *arc associated with the angle.* An analagous series of definitions can be made using clockwise direction. From this point forward, we will use the convention that an angle measured in the clockwise direction from the positive x-axis will have negative measure and be referred to as a **negative angle**, while an angle measured in the counter-clockwise direction from the positive x-axis will have a positive measure and be referred to as a **positive angle**.

Problem 26 *Locate the point on the unit circle so that the angle formed by the radius of the circle containing this point and the positive x-axis is* $-3\pi/4$ *radians. What is this measure of this angle in degrees?*

Problem 27 *Given an angle of degree measure -405, determine the radian measure of this angle.*

Question 28 *Prepare an essay addressing the question: Does mathematics have value to society? Defend your answer and state your sources. If you are unsure as to what an essay is, please request my "essay resource kit" that describes an essay and indicates grading guidelines.*

Problem 29 *Determine the coordinates in the xy-plane of each point on the unit circle whose distance from the point (1,0) along the circle in a counter clockwise direction is an integer multiple of* $\pi/4$.

Problem 30 *Determine the coordinates in the xy-plane of each point on the unit circle such that the angle that is in standard postion with terminal side containing the point has radian measure an integer multiple of* $\pi/6$.

Definition 31 *A* **function** *is a collection of points in the plane with the property that no two of these points lie in the same vertical line.*

You have probably seen the concept of a function denoted by such algebraic expressions as perhaps $f(x) = x^2$ or $y = x^2$. These are not functions, rather they are expressions that represent a function. The function itself is the collection of points (or ordered pairs) that you might graph to form a graphical representation of the function. Thus, I would say that we *define* a function f by the equation $f(x) = x^2$, and it is understood that f is the function consisting of the ordered pairs generated by the equation. Thus $f = \{(1,1), (2,4), (-3,9), \ldots\}$.

Definition 32 *The first coordinates (x-coordinates) of all ordered pairs of a function f are referred to as the* **domain of f** *while the second coordinates (y-coordinates) are referred to as the* **range of f.**

In the next definitions we will use the same type of notation to define a function C as a collection of ordered pairs which are determined by a rule.

Definition 33 *We define the function,* **C**, *so that if $P = (x, y)$ is the point on the unit circle so that the radius of the circle that contains P forms an angle of radian measure θ with the positive x-axis then $C(\theta) = x$. Hence $(\theta, x) = (\theta, C(\theta))$ is a point of the function, C.*

Problem 34 *Determine values for $C(0), C(\pi/6), C(\pi/4), C(\pi/3), C(\pi/2), C(2\pi/3), C(3\pi/4), C(5\pi/6), C(\pi), C(7\pi/6), C(5\pi/4), C(4\pi/3), C(3\pi/2), C(5\pi/3), C(7\pi/4), C(11\pi/6),$ and $C(2\pi)$.*

Problem 35 *Graph the function C, using the ordered pairs, $(\theta, C(\theta))$, computed in the previous problem.*

Problem 36 *Find a value for θ where $C(\theta) = 0$. Are there others?*

Problem 37 *Determine an approximate value for $cos(\pi/5)$.*

Problem 38 *List all values for θ where $C(\theta) = 1/2$.*

Problem 39 *List all values for θ where $C(\theta) = \sqrt{2}/2$.*

Note: The previous problem could be stated, "Solve $C(\theta) = \sqrt{2}/2$ for θ."

Definition 40 *We define the function,* **S**, *so that if $P = (x, y)$ is the point on the unit circle so that the radius of the circle that contains P forms an angle of radian measure θ with the positive x-axis then $S(\theta) = y$.*

Problem 41 *Graph the function S.*

Problem 42 *Solve* $S(\theta) = 1$ *for* θ.

Problem 43 *Solve* $S(\theta) = \sqrt{3}/2$ *for* θ.

Problem 44 *Solve* $S(\theta) = -1/2$ *for* θ.

Problem 45 *Let* f *be the function defined by* $f(\theta) = -S(\theta)$ *for every real number* θ. *Graph* f.

Problem 46 *Let* g *be the function defined by* $g(\theta) = C(\theta + \pi/2)$ *for every real number* θ. *Graph* g.

Definition 47 *Let* T *be the function defined by* $T(\theta) = S(\theta)/C(\theta)$ *for every real number* θ.

Problem 48 *For what values of* θ *will* T *be undefined?*

Problem 49 *Write down a set that is the domain of* T.

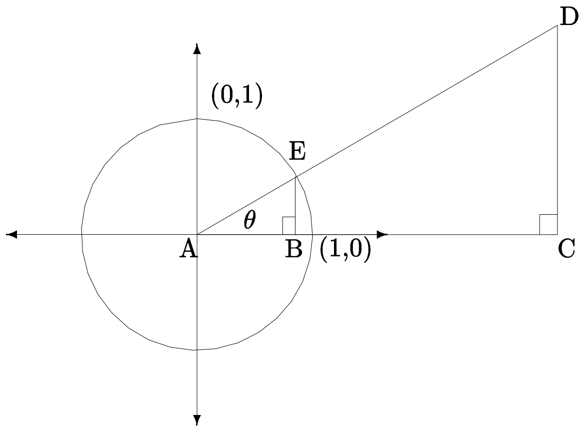
Problem 50 *Graph* T.

Problem 51 *Write an essay on any mathematician's contribution to society. Explain tje contribution in words that your classmates can understand. Make enough copies (with your name omitted if you desire) for your classmates.*

Notation: Given two points, A and B, in the plane, we denote the **line segment** between A and B by **AB** and the length of the line segment by **l(AB)**.

Problem 52 *Let* θ *be a number such that* $0 < \theta < \pi/2$. *Draw the unit circle, the line that is tangent to the unit circle at (1,0), and the line that forms an angle of radian measure* θ *with the positive x-axis.*

Problem 53 *Refer to the picture from the previous problem. Determine the length of the line segment between (1,0) and the intersection of the two lines.*



Note: Refer to the figure above for the next three problems. Assume that the circle is a unit circle, θ is the radian measure of the angle EAB, and $0 < \theta < \pi/2$.

Problem 54 *Show that* $S(\theta) = \frac{l(CD)}{l(AD)}$.

Problem 55 *Show that* $C(\theta) = \frac{l(AC)}{l(AD)}$.

Problem 56 *Show that* $T(\theta) = \frac{l(CD)}{l(AC)}$.

We have now defined three functions, referred to as S, C, and T. We have also shown that if we have a right triangle with one angle of radian measure θ and we assume the side adjacent to this angle has length a, the side opposite from this angle has length o, and the remaining side (the hypotenuse) has length h then these functions satisfy, $S(\theta) = o/h, C(\theta) = a/h$, and $T(\theta) = o/a$. Of course, these are the three trigonometric functions, sine, cosine, and tangent that are commonly abbreviated as sin, cos, and tan respectively. Notice that we also showed that the tangent function gets its name from the fact that it represents the length of a line segment associated with a certain line tangent to the unit circle. We now define three more functions in terms of the sine, cosine, and tangent functions.

Definition 57 *For any number θ for which cosine is non-zero, let secant be the function defined by $sec(\theta) = 1/cos(\theta)$.*

Definition 58 *For any number θ for which sine is non-zero, let cosecant be the function defined by $csc(\theta) = 1/sin(\theta)$.*

Definition 59 *For any number θ for which cosine is non-zero, let cotangent be the function defined by $cot(\theta) = cos(\theta)/sin(\theta)$.*

Problem 60 *Graph the secant function and list the domain and range.*

Problem 61 *Find all numbers u that satisfy,* $2sin(u) = 1$.

Note: The previous problem could be written, "Solve $2sin(u) = 1$ for u."

Problem 62 *Graph the function defined by* $t(x) = 3sin(x - \pi)$.

Problem 63 *Graph the function defined by* $f(x) = 5 - cos(2x)$.

Problem 64 *Graph the function defined by* $r(x) = sin(x) + cos(x)$.

Problem 65 *Graph the function defined by* $z(u) = usin(u)$.

Problem 66 *At what minimum height above ground level must I place a satellite dish so that at a 30 degree angle it will be able to "see" the sky over the top of a building that is 40 feet tall and 50 feet away from the dish.*

Problem 67 *Solve* $2cos(\theta) = -\sqrt{3}$ *for* θ.

Problem 68 *Solve* $tan(\theta) = 1$ *for* θ.

Notation: $sin^n(\theta)$ means $(sin(\theta))^n$ for all values of n except $n = -1$. In the case of $n = -1$ this expression represents the inverse sine function to be defined later. The same rule applies for all six of the trigonometric functions.

Problem 69 *Solve* $cos^2(\theta) - 1 = 0$ *for* θ.

Problem 70 *Graph the cosecant function.*

Problem 71 *Solve* $sin^2(x) - sin(x) = 0$ *for* x.

Problem 72 *Solve* $2sin^2(x/2) - 3sin(x/2) + 1 = 0$ *for* x.

Problem 73 *Krista is standing at the edge of a long straight beach when she sees Jared drowning. Assume that Jared is at a distance of 76 meters straight out from a point on the beach that is 380 meters from where Krista is standing. Assume that Krista can run at 6.5 meters per second and swim at 1.4 meters per second. Krista runs down the beach toward Jared to a point P on the beach and then dives into the water and swims to Jared. The angle at the point P between the beach and the line from P to Jared has measure 77 degrees. How long does it take Krista to save Jared? Could she have saved him faster by taking a different path?*

Problem 74 *Graph cotangent and list its domain and range.*

Problem 75 *Solve* $cot(\theta) > 0$ *for* θ.

Problem 76 *Graph the function defined by* $f(x) = 3 - 2sin(2x + \pi)$.

Problem 77 *Write an essay addressing how you expect to use mathematics upon leaving the university.*

Problem 78 *Let* S *be a square, let* M *and* N *be the midpoints of two adjacent sides, and let* V *be the corner of the square that is opposite both* M *and* N. *Let* θ *be the measure of the angle between the two lines connecting* M *and* N *with* V. *Compute* $sin(\theta)$.

Problem 79 *Graph the function defined by* $f(x) = -2 + 4cos(\pi x + \pi/4)$.

Problem 80 *Solve* $2sin(4x) - \sqrt{3} = 0$ *for* x.

Problem 81 *A plane passes directly over your head at an altitude of 500 feet. Two seconds later you observe that the angle of elevation of the plane is 42 degrees. What is the planes average speed over those two seconds?*

Problem 82 *Solve* $2sin(x/3) + 1 = 0$ *for* x.

Problem 83 *In aerial and nautical navigation, 0 degrees represents due north and directions are measured in degrees clockwise from north, so 90 degrees is due east. If a plane travels 200 miles at a bearing of 300 degrees, how far west of the airport is the plane? How far north? What are the coordinates of the plane?*

Problem 84 *Solve* $4cos^2(2x) - 4cos(2x) + 1 = 0$ *for* x.

Definition 85 *Let* f *be a function and* A *be a positive number. We say that* f *has* **period A** *if* $f(x + A) = f(x)$ *for all* x *in the domain of* f.

Problem 86 *Determine the periods of the six trigonometric functions.*

Problem 87 *Assume that* θ *is a number so that neither* $sin(\theta)$ *nor* $cos(\theta)$ *is zero. Simplify* $tan(\theta)cot(\theta)$. *Why did we make this assumption about sine and cosine not being zero?*

Problem 88 *Show that* $sin^2(t) + cos^2(t) = 1$ *for any number* t.

Problem 89 *Prove that* $tan^2(\theta) + 1 = sec^2(\theta)$ *is valid for all* θ *where* $cos(\theta) \neq 0$.

Problem 90 *Show that* $1 + cot^2(\theta) = csc^2(\theta)$ *is valid for all* θ *where* $sin(\theta) \neq 0$.

The three identities you have just derived are referred to as the **Pythagorean identities.**

Problem 91 *Solve* $1 + cos(x) = sin(x)$ *for* x.

Problem 92 *Solve* $csc(x) + cot(x) = 1$ *for* x.

Problem 93 *Solve* $cos^2(x) = cos(x) + sin^2(x)$ *for* x.

Problem 94 *Simplify* $\frac{tan(\alpha)}{sec(\alpha)csc(\alpha)}$. *For what values of* α *is this expression defined? For what values of* α *is your simplified expression defined? Are the two expressions equal for all values of* α*?*

Problem 95 *Simplify* $(cos^2(\theta) - 1)(tan^2(\theta) + 1)$. *For what values of* θ *is this expression meaningful?*

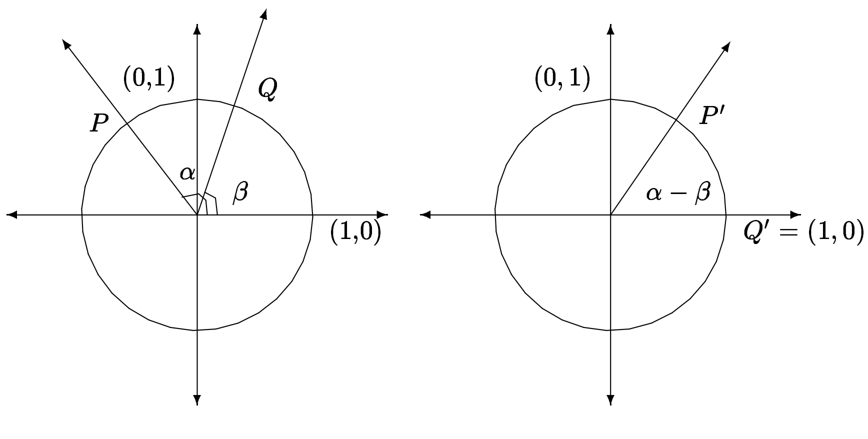
Problem 96 *Simplify* $tan(x)cos(x)$.

Problem 97 *Simplify* $\frac{csc(x)}{sec(x)}$.

Problem 98 *Let* $f(x) = \frac{sec(x)+csc(x)}{1+tan(x)}$. *Simplify* f *and state the domain of* f.

Problem 99 *Simplify* $(csc(x) - 1)(csc(x) + 1)$.

Problem 100 *Simplify* $\frac{1+sec(x)}{tan(x)+sin(x)}$.



Problem 101 *Compute the distance between P and Q in the first illustration.*

Problem 102 *Compute the distance between P' and Q' in the second illustration.*

Problem 103 *Set the results from the two previous problems equal and simplify this expression.*

The expression that you computed in the last problem is referred to as the subtraction identity for the cosine function. This single identity gives rise to a multitude of identities which we will now develop.

Definition 104 *If f is a function and for every number, x, in the domain of f, we have $f(x) = f(-x)$ then we say that f is* **even***.*

Definition 105 *If f is a function and for every number, x, in the domain of f, we have $f(x) = -f(-x)$ then we say that f is* **odd***.*

Problem 106 *Show that every function can be written as the sum of an even and an odd function.*

Note: From a graphical point of view, these definitions correspond to symmetry about the y-axis and symmetry about the origin respectively.

Problem 107 *Let f, g, and h be the functions defined by $f(x) = x^2$, $g(x) = x^3$, and $h(x) = f(x) + g(x)$ for all numbers x. Prove that f is even, g is odd, and h is neither even nor odd.*

Problem 108 *Determine which of the six trigonometric functions are even and which are odd.*

Problem 109 *Prove that if f is an even function and g is an odd function, then the function defined by $h(x) = f(x)g(x)$ is an odd function.*

Problem 110 *Substitute $\alpha = a$ and $\beta = -b$ into the subtraction identity for cosine to develop the addition identity for cosine.*

Problem 111 *Substitute $\alpha = \pi/2$ and $\beta = b$ into the subtraction identity for cosine to develop one of the cofunction (or translation) identities. What does this identity say about the graphs of sine and cosine? List at least three additional cofunction identities.*

Problem 112 *Substitute $\alpha = a$ and $\beta = b - \pi/2$ into the addition identity for cosine to develop yet another identity. What would you call this identity?*

Problem 113 *There are a total of four addition and subtraction identities for sine and cosine and we have developed three. Develop the fourth.*

Problem 114 *Compute a subtraction identity for tangent by simplifying the quotient of the subtraction identity for sine and the subtraction identity for cosine.*

Problem 115 *Compute an addition identity for tangent in a similar manner.*

Problem 116 *Prove or disprove:* $cot(x) + cot(y) = \frac{cos(x-y)}{cos(x)sin(y)}$.

Problem 117 *Prove that* $cot(\pi/2 - x) = tan(x)$.

Problem 118 *Compute an exact value for* $cos(\pi/12)$.

Problem 119 *Compute an exact value for* $sin(7\pi/12)$.

Now we can generate exact values for the sine and cosine of a new set of numbers, specifically, those that are an integer multiple of $\pi/12$. How many angles on the unit circle can we find exact values of sine and cosine for now? What percentage of the total number of numbers can get get an exact value for?

Problem 120 *Prove that* $sin(2x) = 2sin(x)cos(x)$ *is valid for all* x.

Problem 121 *Prove that* $cos(2x) = cos^2(x) - sin^2(x)$ *is valid for all* x.

Problem 122 *Prove that* $tan(2x) = \frac{2tan(x)}{1-tan^2(x)}$ *is valid for all* x.

Problem 123 *Solve* $sin(2x) = sin(x)$ *for* x.

These last three identities are referred to as the **double angle** identities and show up quite a bit in calculus courses. The next two are referred to as the **half angle** identities and can be derived easily from the double angle identities if you can figure out what substition to make.

Problem 124 *Prove that* $sin^2(a/2) = \frac{1-cos(a)}{2}$ *is valid for all* a.

Problem 125 *Prove that* $cos^2(b/2) = \frac{1+cos(b)}{2}$ *is valid for all* b.

Recall that a function is a collection of ordered pairs satisfying a certain property. Not every function has an inverse, but if one does then the inverse of the function is the collection of ordered pairs obtained by reversing the 1^{st} and 2^{nd} coordinate of each ordered pair of f. Thus if f has an inverse and we denote it by f^{-1} then $f(x) = y$ if and only if $f^{-1}(y) = x$. The dilemma is that given a simple function such as $f(x) = x^2$ if we reverse the coordinates then the new set of coordinates do not satisfy the necessary property to be a function. We solve this by restricting the domain. Thus, $f(x) = x^2$ has as its inverse $f^{-1}(x) = \sqrt{x}$ over the domain, $x \geq 0$. The six inverse trigonometric functions are all defined in this manner.

Definition 126 *We define the* **arcsine** *(or inverse sine) to be the function satisfying* $arcsin(x) = y$ *if and only if* $sin(y) = x$ *and having domain* $-1 \leq x \leq 1$ *and range* $-\pi/2 < y < \pi/2$.

Problem 127 *Graph the inverse sine function.*

Definition 128 *We define the* **arccosine** *(or inverse cosine) to be the function satisfying* $arccos(x) = y$ *if and only if* $cos(y) = x$ *and having domain* $-1 \leq x \leq 1$ *and range* $0 < y < \pi$.

Problem 129 *Graph the inverse cosine function.*

Definition 130 *We define the* **arctangent** *(or inverse tangent) to be the function satisfying* $arctan(x) = y$ *if and only if* $tan(y) = x$ *with domain all reals and range* $-\pi/2 < y < \pi/2$.

Problem 131 *Graph the inverse tangent function.*

Notation: The arcsine, arccosine, and arctangent functions are often denoted by sin^{-1}, cos^{-1}, and tan^{-1}. This is an abuse of notation since $sin^{-1}(x) \neq \frac{1}{sin(x)}$. Hence, $sin^n(\theta)$ means $(sin(\theta))^n$ for all values of n *except* $n = -1$ when it denotes the inverse sine function.

Problem 132 *What is the value for* $arcsin(-\sqrt{3}/2)$?

Problem 133 *Compute* $arctan(1)$.

Problem 134 *What is the difference between the results of the next two problems?*

1. *Solve* $cos(\theta) = 1/2$.
2. *Compute* $cos^{-1}(1/2)$.

Problem 135 *Compute* $arccos(-1/\sqrt{2})$.

Problem 136 *Compute* $sin(tan^{-1}(3/4))$.

Problem 137 *Compute* $cot(arccos(-2/3))$.

Problem 138 *Graph* $f(x) = arcsin(x+3)$ *for* $-4 \leq x \leq 2$.

Problem 139 *Graph* $g(x) = -2cos^{-1}(x/3)$ *for* $-3 \leq x \leq 3$.

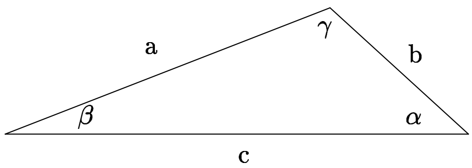
Problem 140 *Graph* $r(x) = arctan(1-2x)$ *for* $-2 \leq x \leq 3$.

Problem 141 *Go to the library. Choose any book on trigonometry and copy the pages that list all the identities. Check to be sure they are correct! Often these can be found on the inside cover of trigonometry or calculus books.*

Theorem 142 *The solutions to* $ax^2 + bx + c = 0$ *are* $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

Problem 143 *Solve* $tan^2(x) = 2tan(x) + 1$ *for x, listing all solutions. Approximate these solutions using your calculator.*

Problem 144 *Solve* $sin^2(x) + 2 = 4sin(x)$ *for x.*



Problem 145 *Viewing the triangle above, show that* $\frac{a}{sin(\alpha)} = \frac{b}{sin(\beta)}$.

Problem 146 *Viewing the triangle above, show that* $a^2 = b^2 + c^2 - 2\ b\ c\ cos(\alpha)$.

For the next few problems, we will adhere to the convention that triangles will be labeled with angles $\alpha, \beta,$ and γ and the sides opposite these angles will have lengths, $a, b,$ and c respectively. The expression, **solve the triangle**, means to provide the lengths of any sides and the measure of any angles that are not supplied in the problem. For each of the following problems, both solve and draw the triangle to a reasonable degree of accuracy.

Problem 147 *Draw and solve any triangles satisfying:* $\alpha = 32, a = 2.5, \beta = 41$.

Problem 148 *Draw and solve any triangles satisfying:* $\alpha = 29, a = 7, c = 14$.

Problem 149 *Draw and solve any triangles satisfying:* $\alpha = 43, b = 6, c = 10$.

Problem 150 *Draw and solve any triangles satisfying:* $\beta = 72.2, b = 78.3, c = 145$.

Problem 151 *Draw and solve any triangles satisfying:* $a = 6, b = 8, c = 9$.

Problem 152 *Draw and solve any triangles satisfying:* $\alpha = 26, a = 11, b = 18$.

Problem 153 *Draw and solve any triangles satisfying:* $a = 8, \beta = 60, c = 11$.

Problem 154 *Draw and solve any triangles satisfying:* $\alpha = 20, b = 10, c = 16$.

Problem 155 *Write an essay describing what, if anything, you have learned from this course that will have a lasting impression on you. Sign, date, and seal this essay. Give me this essay (anonymously, if you wish) after the semester is over.*

Problem 156 *Find the length of one side of a nine-sided regular polygon incribed in a circle of radius 8.32 centimeters.*

Problem 157 *Two birdwatchers, located at points A and B, are twelve and one half miles apart. A Yellow Bellied Sap Sucker is located at point C by both birders. Careful measurements indicate that* $\angle BAC = 14°$ *while* $\angle ABC = 82°$. *Which birder is closer to our Yellow Bellied Sap Sucker and how far is he from the Sap Sucker?*

Problem 158 *An uptight observer stands at ground level some unknown distance away from the base of a building at point A and measures the angle between ground level and the top of the building (called the angle of elevation) to be* $63°$. *After taking this measurement, she walks 140 feet directly away from the building to point B (also at ground level) where she measures the angle of elevation to be* $55°$. *Being weak in trigonometry, she gives you this data. Find the height of the building.*

Note: The following problem was brought to my attention by a very seasoned sailor. He had a Captain's license and regularly commanded vessels with length in excess of 100 feet. He gave the example of a lighthouse that was there to alert sailors to a reef that was 200 yards off the point where the lighthouse was located. However, whenever he sailed there, we could not discern how far off shore he was. He was familiar with angles and navigation utilizing trigonometry, but he was not familiar with the law of sines. To solve his problem we must learn a bit more about nautical and aerial navigation. A direction such as $N30°E$ is read as the direction $30°$ East of due North. Thus in terms of the unit circle this is the direction determined by $\pi/3$. $S40°W$ is $40°$ West of South and corresponds to $230°$ on the unit circle or $220°$ if we view $0°$ as due North.

Problem 159 *A sailor spots a lighthouse at* $N28°E$ *and then proceeds east 7.5 nautical miles where he sites the lighthouse at* $N16°E$*. Find the distance from the boat to the lighthouse. If the boat continues along the same path, determine the minimum distance between the boat and the lighthouse.*

Problem 160 *In order to seal an oil pipeline, we must make a plate of 1/4" steel to place on a flange at the open end of an open pipe. Our plate must be circular with radius 6" and must have 7 holes drilled in it. These holes must be 3/16" in diameter, must be equally spaced, and their centers must be l" from the perimeter of the plate. What should be the distance between the centers of two adjacent holes? Note: Industry standards require an answer accurate to one ten thousandth of an inch.*

Problem 161 *Jack and Jill take off from the same airport at the same time in their new Cesna and Beechcraft planes. Jack flies* $N35°W$ *at 160 miles per hour (mph) while Jill flies* $S70°W$ *at 170 mph. How far apart are the planes after two hours? Determine a function that gives the distance between the planes as a function of time, t, of hours of flight.*

Problem 162 *Two joggers in Central Park are resting on park benches at points A and B, where point A is 1.2 miles north of point B. At midnight both spot a walker. The jogger at point A observes the walker at a heading of* $N20.0°E$ *while the jogger at point B observes the walker at a heading of* $S70°E$*. At 12:10 A.M. the jogger at point A observes the walker at* $N34.0°E$ *and the jogger at point B observes her at* $S55°E$*. Find the walker's average speed.*

Problem 163 *Write an essay on trigonometry.*

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